

PSEUDORIEMANNIAN 2-STEP NILPOTENT LIE GROUPS

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Abstract: We begin a systematic study of these spaces, initially following along the lines of Eberlein's comprehensive study of the Riemannian case. In particular, we integrate the geodesic equation, discuss the structure of the isometry group, and make a study of lattices and periodic geodesics.

Some major differences from the Riemannian theory appear. There are many flat groups (*versus* none), including Heisenberg groups. While still a semidirect product, the isometry group can be strictly larger than the obvious analogue. Everything is illustrated with explicit examples.

We introduce the notion of pH -type, which refines Kaplan's H -type and completes Ciatti's partial extension. We give a general construction for algebras of pH -type.

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1 Introduction

While there had not been much published on the geometry of nilpotent Lie groups with a left-invariant Riemannian metric in 1990 [10], the situation is certainly better now; see [15, 17, 23] and the references in [11]. However, there is still almost nothing extant about the more general pseudoriemannian case. In particular, the 2-step nilpotent groups are nonabelian and as close as possible to being Abelian, but display a rich variety of new and interesting geometric phenomena. As in the Riemannian case, one of many places where they arise naturally is as groups of isometries acting on horospheres in certain (pseudoriemannian) symmetric spaces. Another is in the Iwasawa decomposition of semisimple groups with the Killing metric, which need not be definite. Here we begin the study of these groups.

One motivation for our study was our observation in [8] that there are two nonisometric pseudoriemannian metrics on the Heisenberg group H_3 , one of which is flat. This is a strong contrast to the Riemannian case in which there is only one (up to positive homothety) and it is *not* flat. This is not an anomaly, as we shall see later. We were also inspired by the paper of Eberlein [10, 11], and followed it quite closely in some places. Since the published version is not identical to the preprint, we have cited both where appropriate.

While the geometric properties of Lie groups with left-invariant definite metric tensors have been studied extensively, the same has not occurred for indefinite metric tensors. For example, while the paper of Milnor [18] has already become a classic reference, in particular for the classification of positive definite (Riemannian) metrics on 3-dimensional Lie groups, a classification of the left-invariant Lorentzian metric tensors on these groups became available only very recently [8]. Similarly, only a few partial results in the line of Milnor's study of definite metrics were previously known for indefinite metrics [1, 19]. Moreover, in dimension 3 there are only two types of metric tensors: Riemannian (definite) and Lorentzian (indefinite). But in higher dimensions there are many distinct types of indefinite metrics while there is still essentially only one type of definite metric. This is another reason our work here has special interest.

By an *inner product* on a vector space V we shall mean a nondegenerate, symmetric bilinear form on V , generally denoted by $\langle \cdot, \cdot \rangle$. In particular, we *do not* assume that it is positive definite. Our convention is that $v \in V$ is timelike if $\langle v, v \rangle > 0$, null if $\langle v, v \rangle = 0$, and spacelike if $\langle v, v \rangle < 0$.

Throughout, N will denote a connected, 2-step nilpotent Lie group with Lie algebra \mathfrak{n} having center \mathfrak{z} . (Recall that 2-step means $[\mathfrak{n}, \mathfrak{n}] \subseteq \mathfrak{z}$.) We shall use \langle, \rangle to denote either an inner product on \mathfrak{n} or the induced left-invariant pseudoriemannian (indefinite) metric tensor on N .

In Section 2, we give the fundamental definitions and examples used in the rest of this paper. The main problem encountered is that the center \mathfrak{z} of \mathfrak{n} may be degenerate: it might contain a (totally) null subspace. We shall see that this possible degeneracy of the center causes the essential differences between the Riemannian and pseudoriemannian cases. At the end is our pseudoriemannian pH -type, and a kind of equivalence with H -type. Our pH -type actually refines H -type: each group of H -type has many different (inequivalent) structures of pH -type on it.

Section 3 contains the formulas for the connection and curvatures, and gives the explicit forms in each of the examples from Section 2. We find a relatively large class of flat spaces, a clear distinction from the Riemannian case in which there are none. We also show that a pseudoeuclidean de Rham factor is characterized in terms of j when the center is nondegenerate.

Much like the Riemannian case, we would expect that (N, \langle, \rangle) should in some sense be similar to flat pseudoeuclidean space. This is seen in the examples of totally geodesic subgroups in Section 4. We also show the existence of $\dim \mathfrak{z}$ independent first integrals, a familiar result in pseudoeuclidean space, and integrate the geodesic equations, in certain cases obtaining completely explicit formulas. Unlike the Riemannian case, there are flat groups which are isometric to pseudoeuclidean spaces.

Section 5 contains basic information on the isometry group. In particular, it can be strictly larger in a significant way than in the Riemannian (or pseudoriemannian with nondegenerate center) case.

Section 6 begins with a basic treatment of lattices Γ in these groups. The tori T_F and T_B provide the model fiber and the base for a submersion of $\Gamma \backslash N$. This submersion may not be pseudoriemannian in the usual sense, because the tori may be degenerate. We then begin the study of periodic geodesics in these compact nilmanifolds, obtaining a complete calculation of the period spectrum for the flat spaces of Section 3.

Finally, section 7 details the construction of Lie algebras of pH -type from vector spaces with given data (an inner product and a j -operator). This is a generalization of the construction of Kaplan [13], but we have not used the language of Clifford modules here.

We recall some basic facts about 2-step nilpotent Lie groups. As with all nilpotent Lie groups, the exponential map $\exp : \mathfrak{n} \rightarrow N$ is surjective.

Indeed, it is a diffeomorphism for simply connected N ; in this case we shall denote the inverse by \log . The Baker-Campbell-Hausdorff formula takes on a particularly simple form in these groups:

$$\exp(x)\exp(y) = \exp\left(x + y + \frac{1}{2}[x, y]\right). \quad (1.1)$$

Letting L_n denote left translation by $n \in N$, we have the following description.

Lemma 1.1 *Let \mathfrak{n} denote a 2-step nilpotent Lie algebra and N the corresponding simply connected Lie group. If $x, a \in \mathfrak{n}$, then*

$$\exp_{x*}(a_x) = L_{\exp(x)*}\left(a + \frac{1}{2}[a, x]\right)$$

where a_x denotes the initial velocity vector of the curve $t \mapsto x + ta$. \square

Corollary 1.2 *In a pseudoriemannian 2-step nilpotent Lie group, the exponential map preserves causal character. Alternatively, 1-parameter subgroups are curves of constant causal character.*

Proof: For the 1-parameter subgroup $c(t) = \exp(ta)$, $\dot{c}(t) = \exp_{ta*}(a) = L_{\exp(ta)*}a$ and left translations are isometries. \square

Of course, 1-parameter subgroups need not be geodesics; see Example 4.1.

We shall also need some basic facts about lattices in N . In nilpotent Lie groups, a lattice is a discrete subgroup Γ such that the homogeneous space $M = \Gamma \backslash N$ is compact [22]. Lattices do not always exist in nilpotent Lie groups [16].

Theorem 1.3 *The simply connected, nilpotent Lie group N admits a lattice if and only if there exists a basis of its Lie algebra \mathfrak{n} for which the structure constants are rational.*

Such a group is said to have a rational structure, or simply to be rational.

We recall the result of Marsden from [20].

Theorem 1.4 *A compact, homogeneous pseudoriemannian space is complete.*

Thus if a rational N is provided with a bi-invariant metric tensor $\langle \cdot, \cdot \rangle$, then M becomes a compact, homogeneous pseudoriemannian space which is therefore complete. It follows that $(N, \langle \cdot, \cdot \rangle)$ is itself complete. In general, however, the metric tensor is not bi-invariant and N need not be complete.

For 2-step nilpotent Lie groups, things work nicely as shown by this result first published by Guediri [12].

Theorem 1.5 *On a 2-step nilpotent Lie group, all left-invariant pseudoriemannian metrics are geodesically complete.*

Proof: This follows from the complete integrability of the geodesic equations (4.1), or from the integration of them in Theorem 4.11. \square

He also provided an explicit example of an incomplete metric on a 3-step nilpotent Lie group.

2 Definitions and Examples

In the Riemannian (positive-definite) case, one splits $\mathfrak{n} = \mathfrak{z} \oplus \mathfrak{v} = \mathfrak{z} \oplus \mathfrak{z}^\perp$ where the superscript denotes the orthogonal complement with respect to the inner product \langle, \rangle . In the general pseudoriemannian case, however, $\mathfrak{z} \oplus \mathfrak{z}^\perp \neq \mathfrak{n}$. The problem is that \mathfrak{z} might be a degenerate subspace; *i.e.*, it might contain a null subspace \mathfrak{U} for which $\mathfrak{U} \subseteq \mathfrak{U}^\perp$.

Thus we shall have to adopt a more complicated decomposition of \mathfrak{n} . Observe that if \mathfrak{z} is degenerate, the null subspace \mathfrak{U} is well defined invariantly. We shall use a decomposition

$$\mathfrak{n} = \mathfrak{z} \oplus \mathfrak{v} = \mathfrak{U} \oplus \mathfrak{Z} \oplus \mathfrak{V} \oplus \mathfrak{E}$$

in which $\mathfrak{z} = \mathfrak{U} \oplus \mathfrak{Z}$ and $\mathfrak{v} = \mathfrak{V} \oplus \mathfrak{E}$, \mathfrak{U} and \mathfrak{V} are complementary null subspaces, and $\mathfrak{U}^\perp \cap \mathfrak{V}^\perp = \mathfrak{Z} \oplus \mathfrak{E}$. Although the choice of \mathfrak{V} is *not* well defined invariantly, once a \mathfrak{V} has been chosen then \mathfrak{Z} and \mathfrak{E} are well defined invariantly. Indeed, \mathfrak{Z} is the portion of the center \mathfrak{z} in $\mathfrak{U}^\perp \cap \mathfrak{V}^\perp$ and \mathfrak{E} is its orthocomplement in $\mathfrak{U}^\perp \cap \mathfrak{V}^\perp$.

We now fix a choice of \mathfrak{V} (and therefore \mathfrak{Z} and \mathfrak{E}) to be maintained throughout this paper. Whenever the effect of this choice is to be considered, it will be done so explicitly.

Having fixed \mathfrak{V} , observe that the inner product \langle, \rangle provides a dual pairing between \mathfrak{U} and \mathfrak{V} ; *i.e.*, isomorphisms $\mathfrak{U}^* \cong \mathfrak{V}$ and $\mathfrak{U} \cong \mathfrak{V}^*$. Thus the choice of a basis $\{u_i\}$ in \mathfrak{U} determines an isomorphism $\mathfrak{U} \cong \mathfrak{V}$ (*via* the dual basis $\{v_i\}$ in \mathfrak{V}).

In addition to the choice of \mathfrak{V} , we now also fix a basis of \mathfrak{U} to be maintained throughout this paper. Whenever the effect of this choice is to be considered, it will also be done so explicitly.

We shall also need to use an involution ι that interchanges \mathfrak{U} and \mathfrak{V} by this isomorphism and which reduces to the identity on $\mathfrak{Z} \oplus \mathfrak{E}$ in the

Riemannian (positive-definite) case. The choice of such an involution is not significant, and we have chosen the one which is most natural for the discussion of H -type (see *infra*). In terms of chosen orthonormal bases $\{z_\alpha\}$ of \mathfrak{Z} and $\{e_a\}$ of \mathfrak{E} ,

$$\iota(u_i) = v_i, \quad \iota(v_i) = u_i, \quad \iota(z_\alpha) = \varepsilon_\alpha z_\alpha, \quad \iota(e_a) = \bar{\varepsilon}_a e_a,$$

where, as usual,

$$\langle u_i, v_i \rangle = 1, \quad \langle z_\alpha, z_\alpha \rangle = \varepsilon_\alpha, \quad \langle e_a, e_a \rangle = \bar{\varepsilon}_a.$$

Then $\iota(\mathfrak{U}) = \mathfrak{V}$, $\iota(\mathfrak{V}) = \mathfrak{U}$, $\iota(\mathfrak{Z}) = \mathfrak{Z}$, $\iota(\mathfrak{E}) = \mathfrak{E}$ and $\iota^2 = I$. With respect to this basis $\{u_i, z_\alpha, v_i, e_a\}$ of \mathfrak{n} , ι is given by the following matrix:

$$\begin{bmatrix} & & & 1 \cdots 0 \\ & 0 & 0 & \vdots \cdots \vdots & 0 \\ & & & 0 \cdots 1 \\ & & \varepsilon_1 \cdots 0 \\ & 0 & \vdots \cdots \vdots & 0 & 0 \\ & & 0 \cdots \varepsilon_r \\ 1 \cdots 0 \\ \vdots \cdots \vdots & 0 & 0 & 0 \\ 0 \cdots 1 \\ & & & \bar{\varepsilon}_1 \cdots 0 \\ & 0 & 0 & 0 & \vdots \cdots \vdots \\ & & & 0 \cdots \bar{\varepsilon}_s \end{bmatrix}$$

We note that this is also the matrix of \langle, \rangle on the same basis; however, ι is a linear transformation, so it will transform differently with respect to a change of basis.

It is obvious that ι is self adjoint with respect to the inner product,

$$\langle \iota x, y \rangle = \langle x, \iota y \rangle, \quad x, y \in \mathfrak{n}, \quad (2.1)$$

so ι is an isometry of \mathfrak{n} . (However, it does not integrate to an isometry of N ; see after Example 3.22.) Moreover,

$$\langle x, \iota x \rangle = 0 \text{ if and only if } x = 0, \quad x \in \mathfrak{n}. \quad (2.2)$$

Now consider the adjoint with respect to \langle, \rangle of the adjoint representation of the Lie algebra \mathfrak{n} on itself, to be denoted by ad^\dagger . First note that for all $a \in \mathfrak{Z}$, $\text{ad}_a^\dagger = 0$. Thus for all $y \in \mathfrak{n}$, $\text{ad}_\bullet^\dagger y$ maps $\mathfrak{V} \oplus \mathfrak{E}$ to $\mathfrak{U} \oplus \mathfrak{E}$. Moreover, for all $u \in \mathfrak{U}$ we have $\text{ad}_\bullet^\dagger u = 0$ and for all $e \in \mathfrak{E}$ also $\text{ad}_\bullet^\dagger e = 0$. Following [13, 10, 11], we next define the operator j . Note the use of the involution ι to obtain a good analogy to the Riemannian case.

Definition 2.1 *The linear mapping*

$$j : \mathfrak{U} \oplus \mathfrak{Z} \rightarrow \text{End}(\mathfrak{V} \oplus \mathfrak{E})$$

is given by

$$j(a)x = \iota \text{ad}_x^\dagger \iota a.$$

Equivalently, one has the following characterization:

$$\langle j(a)x, \iota y \rangle = \langle [x, y], \iota a \rangle, \quad a \in \mathfrak{U} \oplus \mathfrak{Z}, \quad x, y \in \mathfrak{V} \oplus \mathfrak{E}. \quad (2.3)$$

We shall see in Section 3 that this map (together with the Lie algebra structure of \mathfrak{n}) determines the geometry of N just as in the Riemannian case [11].

The operator j is ι -skewadjoint with respect to the inner product $\langle \cdot, \cdot \rangle$.

Proposition 2.2 *For every $a \in \mathfrak{U} \oplus \mathfrak{Z}$ and all $x, y \in \mathfrak{V} \oplus \mathfrak{E}$,*

$$\langle j(a)x, \iota y \rangle + \langle \iota x, j(a)y \rangle = 0.$$

Proof:

$$\langle j(a)x, \iota y \rangle = \langle [x, y], \iota a \rangle = -\langle [y, x], \iota a \rangle = -\langle \iota x, j(a)y \rangle \quad \square$$

Although this is not the traditional skewadjoint property, it turns out to be just what is needed in our generalization of H -type (cf. Definition 2.8 and following). The Riemannian version [13, 11] is easily obtained as the particular case when we assume $\mathfrak{U} = \mathfrak{V} = \{0\}$ and the inner product on $\mathfrak{Z} \oplus \mathfrak{E}$ is positive definite (i.e., $\varepsilon_\alpha = \bar{\varepsilon}_\alpha = 1$); then $\iota = I$ and we recover the definitions of [13]. This recovery of the Riemannian case continues in all that follows.

Recall that the Lie algebra \mathfrak{n} is said to be nonsingular if and only if ad_x maps \mathfrak{n} onto \mathfrak{z} for every $x \in \mathfrak{n} - \mathfrak{z}$. As in [11], we immediately obtain

Lemma 2.3 *The 2-step nilpotent Lie algebra \mathfrak{n} is nonsingular if and only if for every $a \in \mathfrak{U} \oplus \mathfrak{Z}$ the maps $j(a)$ are nonsingular, for every inner product on \mathfrak{n} , every choice of \mathfrak{V} , every basis of \mathfrak{U} , and every choice of ι .* \square

Next we consider some examples of these Lie groups to which we shall return in subsequent sections.

Example 2.4 The usual inner products on the 3-dimensional Heisenberg algebra \mathfrak{h}_3 may be described as follows. On an orthonormal basis $\{z, e_1, e_2\}$ the structure equation is

$$[e_1, e_2] = z$$

with nontrivial inner products

$$\varepsilon = \langle z, z \rangle, \quad \bar{\varepsilon}_1 = \langle e_1, e_1 \rangle, \quad \bar{\varepsilon}_2 = \langle e_2, e_2 \rangle.$$

We find the nontrivial adjoint maps as

$$\text{ad}_{e_1}^\dagger z = \varepsilon \bar{\varepsilon}_2 e_2, \quad \text{ad}_{e_2}^\dagger z = -\varepsilon \bar{\varepsilon}_1 e_1$$

with the rest vanishing. On the basis $\{e_1, e_2\}$,

$$j(z) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

so $j(z)^2 = -I_2$. Moreover, a direct computation shows that

$$j(a)^2 = -\langle a, \iota a \rangle I_2, \quad \text{for all } a \in \mathfrak{Z}.$$

This construction extends to the generalized Heisenberg groups $H(p, 1)$ of dimension $2p + 1$ with non-null center of dimension 1 generated by z , and we find

$$j(z) = \begin{bmatrix} 0 & -I_p \\ I_p & 0 \end{bmatrix}.$$

Example 2.5 In [8] we gave an inner product on the 3-dimensional Heisenberg algebra \mathfrak{h}_3 for which the center is degenerate. First we recall that on an orthonormal basis $\{e_1, e_2, e_3\}$ with signature $(+ - -)$ the structure equations are

$$\begin{aligned} [e_3, e_1] &= \frac{1}{2}(e_1 - e_2), \\ [e_3, e_2] &= \frac{1}{2}(e_1 - e_2), \\ [e_1, e_2] &= 0. \end{aligned}$$

The center \mathfrak{z} is the span of $e_1 - e_2$ and is in fact null.

We take a new basis $\{u = \frac{1}{\sqrt{2}}(e_2 - e_1), v = \frac{1}{\sqrt{2}}(e_2 + e_1), e = e_3\}$ and find the structure equation

$$[v, e] = u.$$

Generalizing slightly, we take the nontrivial inner products to be

$$\langle u, v \rangle = 1 \quad \text{and} \quad \langle e, e \rangle = \bar{\varepsilon}.$$

We find the nontrivial adjoint maps as

$$\text{ad}_v^\dagger v = \bar{\varepsilon} e, \quad \text{ad}_e^\dagger v = -u.$$

On the basis $\{v, e\}$,

$$j(u) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

so $j(u)^2 = -I_2$. Again, a direct computation shows that $j(a)^2 = -\langle a, \iota a \rangle I_2$ for all $a \in \mathfrak{U}$.

Again, this construction extends to $H(p, 1)$ with null center generated by u and we find

$$j(u) = \begin{bmatrix} 0 & -I_p \\ I_p & 0 \end{bmatrix}.$$

According to [8], these are all the Lorentzian inner products on \mathfrak{h}_3 up to homothety.

Example 2.6 For the simplest quaternionic Heisenberg algebra of dimension 7, we may take a basis $\{u_1, u_2, z, v_1, v_2, e_1, e_2\}$ with structure equations

$$\begin{aligned} [e_1, e_2] &= z & [v_1, v_2] &= z \\ [e_1, v_1] &= u_1 & [e_2, v_1] &= u_2 \\ [e_1, v_2] &= u_2 & [e_2, v_2] &= -u_1 \end{aligned}$$

and nontrivial inner products

$$\langle u_i, v_j \rangle = \delta_{ij}, \quad \langle z, z \rangle = \varepsilon, \quad \langle e_a, e_a \rangle = \bar{\varepsilon}_a.$$

As usual, each ε -symbol is ± 1 independently (this is a combined null and orthonormal basis), so the signature is $(++--\varepsilon\bar{\varepsilon}_1\bar{\varepsilon}_2)$.

The nontrivial adjoint maps are

$$\begin{aligned} \text{ad}_{v_1}^\dagger z &= \varepsilon u_2 & \text{ad}_{e_1}^\dagger z &= \varepsilon \bar{\varepsilon}_2 e_2 \\ \text{ad}_{v_1}^\dagger v_1 &= -\bar{\varepsilon}_1 e_1 & \text{ad}_{e_1}^\dagger v_1 &= u_1 \\ \text{ad}_{v_1}^\dagger v_2 &= -\bar{\varepsilon}_2 e_2 & \text{ad}_{e_1}^\dagger v_2 &= u_2 \\ \text{ad}_{v_2}^\dagger z &= -\varepsilon u_1 & \text{ad}_{e_2}^\dagger z &= -\varepsilon \bar{\varepsilon}_1 e_1 \\ \text{ad}_{v_2}^\dagger v_1 &= \bar{\varepsilon}_2 e_2 & \text{ad}_{e_2}^\dagger v_1 &= -u_2 \\ \text{ad}_{v_2}^\dagger v_2 &= -\bar{\varepsilon}_1 e_1 & \text{ad}_{e_2}^\dagger v_2 &= u_1 \end{aligned}$$

For j on the basis $\{v_1, v_2, e_1, e_2\}$ we obtain

$$j(u_1) = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

so $j(u_1)^2 = -I_4$,

$$j(u_2) = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}$$

so $j(u_2)^2 = -I_4$, and

$$j(z) = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

so $j(z)^2 = -I_4$. Again, a direct computation shows that $j(a)^2 = -\langle a, \iota a \rangle I_4$ for all $a \in \mathfrak{U} \oplus \mathfrak{Z}$.

Once again, this construction may be extended to a nondegenerate center, to a degenerate center with a null subspace of dimensions 1 or 3, and to quaternionic algebras of all dimensions.

All of the examples so far are of H -type; the next one is not.

Example 2.7 For the generalized Heisenberg group $H(1, 2)$ of dimension 5 we take the basis $\{u, z, v, e_1, e_2\}$ with structure equations

$$\begin{aligned} [e_1, e_2] &= z \\ [v, e_2] &= u \end{aligned}$$

and nontrivial inner products

$$\langle u, v \rangle = 1, \quad \langle z, z \rangle = \varepsilon, \quad \langle e_a, e_a \rangle = \bar{\varepsilon}_a.$$

The signature is $(+ - \varepsilon \bar{\varepsilon}_1 \bar{\varepsilon}_2)$. We find the nontrivial adjoint maps.

$$\begin{aligned} \text{ad}_v^\dagger v &= \bar{\varepsilon}_2 e_2 & \text{ad}_{e_1}^\dagger z &= \varepsilon \bar{\varepsilon}_2 e_2 \\ \text{ad}_{e_2}^\dagger v &= -u & \text{ad}_{e_2}^\dagger z &= -\varepsilon \bar{\varepsilon}_1 e_1 \end{aligned}$$

On the basis $\{v, e_1, e_2\}$,

$$j(u) = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

so $j(u)^2 \cong 0 \oplus -I_2$, and

$$j(z) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$$

with $j(z)^2 = 0 \oplus -I_2$.

This construction, too, extends to the generalized Heisenberg groups $H(1, p)$ with $p \geq 2$ which are not of H -type. The dimension is again $2p + 1$, but now the center has dimension p . In each case, the j -endomorphisms have rank 2 with a similar appearance.

From these examples, it seems natural to extend the usual (positive-definite) definition of H -type to indefinite inner products.

Definition 2.8 *A 2-step nilpotent Lie algebra \mathfrak{n} with indefinite inner product \langle, \rangle and chosen subspace \mathfrak{V} as in Section 2 is said to be of pH -type (pseudo H -type) if and only if*

$$j(a)^2 = -\langle a, \iota a \rangle I \quad \text{on } \mathfrak{V} \oplus \mathfrak{E}$$

for every choice of $a \in \mathfrak{U} \oplus \mathfrak{Z}$.

We note that it is easy to verify that this is true for one choice of \mathfrak{V} and ι if and only if it is true for any choice of \mathfrak{V} and ι . In the positive-definite case, this reduces to the usual notion of H -type [11, 13]. In the case of a nondegenerate center, Ciatti [6] has made an equivalent definition and obtained similar results. Clearly, if \mathfrak{n} is of pH -type then \mathfrak{n} is nonsingular.

Now it is easy to obtain formulas which are similar to those in the positive-definite case (see, for example, [11], (1.7)).

Proposition 2.9 *If $(\mathfrak{n}, \langle, \rangle, \mathfrak{V})$ is of pH -type then the following identities hold:*

$$\begin{aligned} \langle j(a)x, \iota j(a)x \rangle &= \langle a, \iota a \rangle \langle x, \iota x \rangle && \text{for all } a \in \mathfrak{U} \oplus \mathfrak{Z}, x \in \mathfrak{V} \oplus \mathfrak{E}; \\ \langle j(a)x, \iota j(a)y \rangle &= \langle a, \iota a \rangle \langle x, \iota y \rangle && \text{for all } a \in \mathfrak{U} \oplus \mathfrak{Z}, x, y \in \mathfrak{V} \oplus \mathfrak{E}; \\ \langle j(a)x, \iota j(b)x \rangle &= \langle a, \iota b \rangle \langle x, \iota x \rangle && \text{for all } a, b \in \mathfrak{U} \oplus \mathfrak{Z}, x \in \mathfrak{V} \oplus \mathfrak{E}; \\ j(a) \circ j(b) + j(b) \circ j(a) &= -2\langle a, \iota b \rangle I && \text{for all } a, b \in \mathfrak{U} \oplus \mathfrak{Z}. \end{aligned}$$

Proof: All the identities follow easily taking into account (2.1), Proposition 2.2, and the definition of pH -type.

For the first, we have

$$\begin{aligned}\langle j(a)x, \iota j(a)x \rangle &= -\langle \iota x, j(a)(j(a)x) \rangle \\ &= -\langle \iota x, -\langle a, \iota a \rangle x \rangle \\ &= \langle a, \iota a \rangle \langle x, \iota x \rangle.\end{aligned}$$

The second and the third identities follow from this one by polarization. For the second, we compute as follows:

$$\langle j(a)(x+y), \iota j(a)(x+y) \rangle = \langle a, \iota a \rangle \langle x, \iota x \rangle + 2\langle j(a)x, \iota j(a)y \rangle + \langle a, \iota a \rangle \langle y, \iota y \rangle;$$

on the other hand,

$$\begin{aligned}\langle j(a)(x+y), \iota j(a)(x+y) \rangle &= \langle a, \iota a \rangle \langle x+y, \iota(x+y) \rangle \\ &= \langle a, \iota a \rangle (\langle x, \iota x \rangle + 2\langle x, \iota y \rangle + \langle y, \iota y \rangle) .\end{aligned}$$

Similarly for the third:

$$\langle j(a+b)x, \iota j(a+b)x \rangle = \langle a, \iota a \rangle \langle x, \iota x \rangle + 2\langle j(a)x, \iota j(b)x \rangle + \langle b, \iota b \rangle \langle x, \iota x \rangle;$$

on the other hand,

$$\begin{aligned}\langle j(a+b)x, \iota j(a+b)x \rangle &= \langle a+b, \iota(a+b) \rangle \langle x, \iota x \rangle \\ &= (\langle a, \iota a \rangle + 2\langle a, \iota b \rangle + \langle b, \iota b \rangle) \langle x, \iota x \rangle.\end{aligned}$$

In order to prove the fourth identity, we polarize the formula in Definition 2.8:

$$\begin{aligned}j(a+b)^2x &= -\langle a+b, \iota(a+b) \rangle x \\ &= -(\langle a, \iota a \rangle + 2\langle a, \iota b \rangle + \langle b, \iota b \rangle) x \\ &= j(a)^2x - 2\langle a, \iota b \rangle x + j(b)^2x ;\end{aligned}$$

on the other hand,

$$j(a+b)(j(a+b)x) = j(a)^2x + j(a)(j(b)x) + j(b)(j(a)x) + j(b)^2x . \quad \square$$

There are no new 2-step nilpotent groups of pH -type.

Proposition 2.10 *The 2-step nilpotent Lie group N is of H -type if and only if it is of pH -type.*

Proof: Suppose $(N, \langle, \rangle, \mathfrak{V}, \iota)$ is of pH -type. Consider the associated left-invariant Riemannian metric $(,)$ given by $(x, y) = \langle x, \iota y \rangle$ and let j and \tilde{j} be the respective j -maps. Then $(\tilde{j}(a)x, y) = ([x, y], a) = \langle [x, y], \iota a \rangle = \langle j(a)x, \iota y \rangle$ for all $a \in \mathfrak{z}$ and $x, y \in \mathfrak{v}$, noting that \mathfrak{v} is the same for both inner products by construction. Thus $j(a) = \tilde{j}(a)$ for every $a \in \mathfrak{z}$, so $\tilde{j}(a)^2 = j(a)^2 = -\langle a, \iota a \rangle I = -(a, a)I$ and $(N, (,))$ is of H -type.

The converse follows by choosing any $(,)$ -orthonormal basis, then choosing an associated \langle, \rangle and ι , and then reversing the preceding argument. \square

But there are a *lot* of new geometries now on these groups. Indeed, given a left-invariant Riemannian metric $(,)$ on a Lie group N of H -type, the number of nonisometric, associated pseudoriemannian metrics \langle, \rangle grows exponentially with $\dim N$, as is easily seen by looking at the possible matrices for ι .

Thus the notion of H -type or pH -type is more fundamentally a group property than a geometric property. This was already suggested by the result of Kaplan [14] that a naturally reductive group of H -type has a center of dimension 1 or 3 only. Our result also shows that Ciatti's [6] family of admissible algebras with nondegenerate center is a subfamily of ours, and thus part of Kaplan's [13]. See Section 7 for a general construction of algebras of pH -type.

3 Connection and Curvatures

The Levi-Civita connection is given by

$$\nabla_x y = \frac{1}{2} \left([x, y] - \text{ad}_x^\dagger y - \text{ad}_y^\dagger x \right)$$

for all $x, y \in \mathfrak{n}$.

Theorem 3.1 *Let N be a 2-step nilpotent Lie group with left-invariant pseudoriemannian metric tensor \langle, \rangle and Lie algebra $\mathfrak{n} = \mathfrak{U} \oplus \mathfrak{Z} \oplus \mathfrak{V} \oplus \mathfrak{E}$ decomposed as in Section 2. For all $u, u' \in \mathfrak{U}$, $z, z' \in \mathfrak{Z}$, $v, v' \in \mathfrak{V}$, $e, e' \in \mathfrak{E}$, and $x \in \mathfrak{V} \oplus \mathfrak{E}$, we have*

$$\nabla_u u' = \nabla_u z = \nabla_z u = \nabla_z z' = 0, \quad (3.1)$$

$$\nabla_u v = \nabla_v u = \nabla_u e = \nabla_e u = 0, \quad (3.2)$$

$$\nabla_z x = \nabla_x z = -\frac{1}{2} \iota j(\iota z) x, \quad (3.3)$$

$$\nabla_v v' = \frac{1}{2} \left([v, v'] - \iota j(\iota v) v' - \iota j(\iota v') v \right), \quad (3.4)$$

$$\nabla_v e = \frac{1}{2} \left([v, e] - \iota j(\iota v) e \right), \quad (3.5)$$

$$\nabla_e v = \frac{1}{2} \left([e, v] - \iota j(\iota v) e \right), \quad (3.6)$$

$$\nabla_e e' = \frac{1}{2} [e, e']. \quad (3.7)$$

□

Example 3.2 Continuing from Example 2.4, the Levi-Civita connection for the Heisenberg group with non-null center is given by

$$\begin{aligned} \nabla_z e_1 &= \nabla_{e_1} z = -\frac{1}{2} \varepsilon \bar{\varepsilon}_2 e_2 & \nabla_{e_1} e_2 &= \frac{1}{2} z = -\nabla_{e_2} e_1 \\ \nabla_z e_2 &= \nabla_{e_2} z = \frac{1}{2} \varepsilon \bar{\varepsilon}_1 e_1 \end{aligned}$$

with the rest vanishing.

Example 3.3 Continuing Example 2.5, the Levi-Civita connection for the Heisenberg group with null center is given by

$$\nabla_v v = -\bar{\varepsilon} e \quad \nabla_v e = u$$

with the rest vanishing.

Example 3.4 Continuing Example 2.6, the Levi-Civita connection for our quaternionic Heisenberg algebra is given by

$$\begin{aligned} \nabla_z v_1 &= \nabla_{v_1} z = -\frac{1}{2} \varepsilon u_2 & \nabla_z e_1 &= \nabla_{e_1} z = -\frac{1}{2} \varepsilon \bar{\varepsilon}_2 e_2 \\ \nabla_z v_2 &= \nabla_{v_2} z = \frac{1}{2} \varepsilon u_1 & \nabla_z e_2 &= \nabla_{e_2} z = \frac{1}{2} \varepsilon \bar{\varepsilon}_1 e_1 \\ \nabla_{v_1} v_1 &= \bar{\varepsilon}_1 e_1 & \nabla_{v_1} e_1 &= -u_1 \\ \nabla_{v_1} v_2 &= \frac{1}{2} z & \nabla_{v_2} e_1 &= -u_2 \\ \nabla_{v_2} v_1 &= -\frac{1}{2} z & \nabla_{e_2} v_1 &= u_2 \\ \nabla_{v_2} v_2 &= \bar{\varepsilon}_1 e_1 & \nabla_{e_2} v_2 &= -u_1 \\ \nabla_{e_1} e_2 &= \frac{1}{2} z = -\nabla_{e_2} e_1 \end{aligned}$$

with the rest vanishing.

Example 3.5 Continuing Example 2.7, the Levi-Civita connection for the group $H(1, 2)$ is given by

$$\begin{aligned} \nabla_z e_1 &= \nabla_{e_1} z = -\frac{1}{2} \varepsilon \bar{\varepsilon}_2 e_2 & \nabla_z e_2 &= \nabla_{e_2} z = \frac{1}{2} \varepsilon \bar{\varepsilon}_1 e_1 \\ \nabla_v v &= -\bar{\varepsilon}_2 e_2 & \nabla_v e_2 &= u \\ \nabla_{e_1} e_2 &= \frac{1}{2} z = -\nabla_{e_2} e_1 \end{aligned}$$

with the rest vanishing.

The curvature operator is given by

$$R(x, y)w = \nabla_x \nabla_y w - \nabla_y \nabla_x w - \nabla_{[x, y]} w$$

for all $x, y, w \in \mathfrak{n}$. We denote the projections of vectors in \mathfrak{n} onto the subspaces of Section 2 by superscripts, $w = w^{\mathfrak{U}} + w^{\mathfrak{Z}} + w^{\mathfrak{V}} + w^{\mathfrak{E}}$, and note that $(\iota w)^{\mathfrak{U}} = \iota w^{\mathfrak{V}}$, $(\iota w)^{\mathfrak{Z}} = \iota w^{\mathfrak{Z}}$, $(\iota w)^{\mathfrak{V}} = \iota w^{\mathfrak{U}}$, and $(\iota w)^{\mathfrak{E}} = \iota w^{\mathfrak{E}}$.

Theorem 3.6 *Let N be a 2-step nilpotent Lie group with left-invariant pseudoriemannian metric tensor $\langle \cdot, \cdot \rangle$ and Lie algebra $\mathfrak{n} = \mathfrak{U} \oplus \mathfrak{Z} \oplus \mathfrak{V} \oplus \mathfrak{E}$ decomposed as in Section 2. For all $u, u' \in \mathfrak{U}$, $z, z' \in \mathfrak{Z}$, $v, v' \in \mathfrak{V}$, $e, e' \in \mathfrak{E}$, and $x \in \mathfrak{V} \oplus \mathfrak{E}$, we have*

$$R(u, \cdot) = R(\cdot, \cdot)u = 0, \quad (3.8)$$

$$R(z, z')z'' = 0, \quad (3.9)$$

$$R(z, z')x = \frac{1}{4}(\iota j(\iota z)[\iota j(\iota z')x]^{\mathfrak{E}} - \iota j(\iota z')[\iota j(\iota z)x]^{\mathfrak{E}}), \quad (3.10)$$

$$R(z, x)z' = \frac{1}{4}\iota j(\iota z)[\iota j(\iota z')x]^{\mathfrak{E}}, \quad (3.11)$$

$$R(z, v)v' = \frac{1}{4}([v, \iota j(\iota z)v'] + \iota j(\iota z)[\iota j(\iota v)v']^{\mathfrak{E}} - \iota j(\iota v)[\iota j(\iota z)v']^{\mathfrak{E}} + \iota j(\iota z)[\iota j(\iota v')v]^{\mathfrak{E}}), \quad (3.12)$$

$$R(z, v)e = \frac{1}{4}([v, \iota j(\iota z)e] + \iota j(\iota z)[\iota j(\iota v)e]^{\mathfrak{E}} - \iota j(\iota v)[\iota j(\iota z)e]^{\mathfrak{E}}), \quad (3.13)$$

$$R(z, e)v = \frac{1}{4}([e, \iota j(\iota z)v] + \iota j(\iota z)[\iota j(\iota v)e]^{\mathfrak{E}}), \quad (3.14)$$

$$R(z, e)e' = \frac{1}{4}[e, \iota j(\iota z)e'], \quad (3.15)$$

$$R(v, v')z = -\frac{1}{4}([\iota j(\iota z)v, v'] + [v, \iota j(\iota z)v'] - \iota j(\iota v)[\iota j(\iota z)v']^{\mathfrak{E}} + \iota j(\iota v')[\iota j(\iota z)v]^{\mathfrak{E}}), \quad (3.16)$$

$$R(v, v')v'' = -\frac{1}{4}([v, \iota j(\iota v')v'' + \iota j(\iota v'')v'] - [v', \iota j(\iota v)v'' + \iota j(\iota v'')v] - \iota j(\iota[v, v'']^{\mathfrak{Z}})v' + \iota j(\iota[v', v'']^{\mathfrak{Z}})v - \iota j(\iota v)[\iota j(\iota v')v'']^{\mathfrak{E}} + \iota j(\iota v')[\iota j(\iota v)v'']^{\mathfrak{E}} - \iota j(\iota v)[\iota j(\iota v'')v']^{\mathfrak{E}} + \iota j(\iota v')[\iota j(\iota v'')v]^{\mathfrak{E}}) + \frac{1}{2}\iota j(\iota[v, v']^{\mathfrak{Z}})v'', \quad (3.17)$$

$$R(v, v')e = -\frac{1}{4}([\iota j(\iota v)e, v'] + [v, \iota j(\iota v')e] - \iota j(\iota[v, e]^{\mathfrak{Z}})v' + \iota j(\iota[v', e]^{\mathfrak{Z}})v - \iota j(\iota v)[\iota j(\iota v')e]^{\mathfrak{E}} + \iota j(\iota v')[\iota j(\iota v)e]^{\mathfrak{E}}) + \frac{1}{2}\iota j(\iota[v, v']^{\mathfrak{Z}})e, \quad (3.18)$$

$$R(v, e)z = -\frac{1}{4}\left([v, \iota j(\iota z)e] + [\iota j(\iota z)v, e] - \iota j(\iota v)[\iota j(\iota z)e]^\mathfrak{e}\right), \quad (3.19)$$

$$\begin{aligned} R(v, e)v' &= -\frac{1}{4}\left([v, \iota j(\iota v')e] + [\iota j(\iota v)v', e] \right. \\ &\quad \left. - \iota j(\iota[v, v']^3)e - \iota j(\iota[v', e]^3)v - \iota j(\iota v)[\iota j(\iota v')e]^\mathfrak{e}\right) \\ &\quad + \frac{1}{2}\iota j(\iota[v, e]^3)v', \end{aligned} \quad (3.20)$$

$$\begin{aligned} R(v, e)e' &= \frac{1}{4}\left([e, \iota j(\iota v)e'] + \iota j(\iota[v, e']^3)e - \iota j(\iota[e, e']^3)v\right) \\ &\quad + \frac{1}{2}\iota j(\iota[v, e]^3)e', \end{aligned} \quad (3.21)$$

$$R(e, e')z = -\frac{1}{4}\left([e, \iota j(\iota z)e'] + [\iota j(\iota z)e, e']\right), \quad (3.22)$$

$$\begin{aligned} R(e, e')v &= -\frac{1}{4}\left([e, \iota j(\iota v)e'] + [\iota j(\iota v)e, e'] + \iota j(\iota[v, e]^3)e' \right. \\ &\quad \left. - \iota j(\iota[v, e']^3)e\right) + \frac{1}{2}\iota j(\iota[e, e']^3)v, \end{aligned} \quad (3.23)$$

$$R(e, e')e'' = \frac{1}{4}\left(\iota j(\iota[e, e'']^3)e' - \iota j(\iota[e', e'']^3)e\right) + \frac{1}{2}\iota j(\iota[e, e']^3)e''. \quad (3.24)$$

□

Together with the formulas for the connection in Theorem 3.1, these show that the map j together with the Lie algebra structure of \mathfrak{n} does indeed completely determine the geometry of N as in the Riemannian case. Since j only occurs as $\iota j \iota = \text{ad}^\dagger$, the appearance of ι is an artifact. It might better be said that the map ad^\dagger and the Lie algebra structure of \mathfrak{n} completely determine the geometry.

Example 3.7 Continuing from Examples 2.4 and 3.2, we find the nontrivial curvature operators for the Heisenberg group with non-null center.

$$\begin{aligned} R(z, e_1)z &= -\frac{1}{4}\bar{\varepsilon}_1\bar{\varepsilon}_2 e_1 & R(z, e_2)z &= -\frac{1}{4}\bar{\varepsilon}_1\bar{\varepsilon}_2 e_2 \\ R(z, e_1)e_1 &= \frac{1}{4}\varepsilon\bar{\varepsilon}_2 z & R(e_1, e_2)e_1 &= \frac{3}{4}\varepsilon\bar{\varepsilon}_2 e_2 \\ R(z, e_2)e_2 &= \frac{1}{4}\varepsilon\bar{\varepsilon}_1 z & R(e_1, e_2)e_2 &= -\frac{3}{4}\varepsilon\bar{\varepsilon}_1 e_1 \end{aligned}$$

Example 3.8 Continuing from Examples 2.5 and 3.3, we find that all curvatures for the Heisenberg group with null center vanish identically: this example is flat.

When $p \geq 2$, however, the geometry on $H(p, 1)$ is *not* flat.

Thus there is more than one geometry on the Heisenberg group up to homothety, in contrast to the Riemannian case of only one.

Example 3.9 Continuing from Examples 2.6 and 3.4, we find the nontrivial curvature operators for our quaternionic Heisenberg algebra.

$$\begin{aligned}
R(z, v_1)v_1 &= -\frac{1}{2}\varepsilon\bar{\varepsilon}_1\bar{\varepsilon}_2 e_2 & R(z, e_1)z &= -\frac{1}{4}\bar{\varepsilon}_1\bar{\varepsilon}_2 e_1 \\
R(z, v_1)e_2 &= \frac{1}{2}\varepsilon\bar{\varepsilon}_1 u_1 & R(z, e_1)e_1 &= \frac{1}{4}\varepsilon\bar{\varepsilon}_2 z \\
R(z, v_2)v_2 &= -\frac{1}{2}\varepsilon\bar{\varepsilon}_1\bar{\varepsilon}_2 e_2 & R(z, e_2)z &= -\frac{1}{4}\bar{\varepsilon}_1\bar{\varepsilon}_2 e_2 \\
R(z, v_2)e_2 &= \frac{1}{2}\varepsilon\bar{\varepsilon}_1 u_2 & R(z, e_2)e_2 &= \frac{1}{4}\varepsilon\bar{\varepsilon}_1 z \\
\\
R(v_1, v_2)v_1 &= (\bar{\varepsilon}_1 + \frac{3}{4}\varepsilon)u_2 & R(v_1, e_1)v_2 &= \frac{1}{4}\varepsilon\bar{\varepsilon}_2 e_2 \\
R(v_1, v_2)v_2 &= -(\bar{\varepsilon}_1 + \frac{3}{4}\varepsilon)u_1 & R(v_1, e_1)e_2 &= -\frac{1}{4}\varepsilon u_2 \\
R(v_1, v_2)e_1 &= \frac{1}{2}\varepsilon\bar{\varepsilon}_2 e_2 & R(v_1, e_2)z &= -\frac{1}{2}\varepsilon\bar{\varepsilon}_1 u_1 \\
R(v_1, v_2)e_2 &= -\frac{1}{2}\varepsilon\bar{\varepsilon}_1 e_1 & R(v_1, e_2)v_1 &= \frac{1}{2}\bar{\varepsilon}_1 z \\
R(v_1, e_2)v_2 &= -\frac{1}{4}\varepsilon\bar{\varepsilon}_1 e_1 & R(v_1, e_2)e_1 &= \frac{1}{4}\varepsilon u_2 \\
\\
R(v_2, e_1)v_1 &= -\frac{1}{4}\varepsilon\bar{\varepsilon}_2 e_2 & R(v_2, e_1)e_2 &= \frac{1}{4}\varepsilon u_1 \\
R(v_2, e_2)z &= -\frac{1}{2}\varepsilon\bar{\varepsilon}_1 u_2 & R(e_1, e_2)v_1 &= \frac{1}{2}\varepsilon u_2 \\
R(v_2, e_2)v_1 &= \frac{1}{4}\varepsilon\bar{\varepsilon}_1 e_1 & R(e_1, e_2)v_2 &= -\frac{1}{2}\varepsilon u_1 \\
R(v_2, e_2)v_2 &= \frac{1}{2}\bar{\varepsilon}_1 z & R(e_1, e_2)e_1 &= \frac{3}{4}\varepsilon\bar{\varepsilon}_2 e_2 \\
R(v_2, e_2)e_1 &= -\frac{1}{4}\varepsilon u_1 & R(e_1, e_2)e_2 &= -\frac{3}{4}\varepsilon\bar{\varepsilon}_1 e_1
\end{aligned}$$

Example 3.10 Continuing from Examples 2.7 and 3.5, we find the nontrivial curvature operators for the group $H(1, 2)$.

$$\begin{aligned}
R(z, v)v &= -\frac{1}{2}\varepsilon\bar{\varepsilon}_1\bar{\varepsilon}_2 e_1 & R(z, v)e_1 &= \frac{1}{2}\varepsilon\bar{\varepsilon}_2 u \\
R(z, e_1)z &= -\frac{1}{4}\bar{\varepsilon}_1\bar{\varepsilon}_2 e_1 & R(z, e_1)e_1 &= \frac{1}{4}\varepsilon\bar{\varepsilon}_2 z \\
R(z, e_2)z &= -\frac{1}{4}\bar{\varepsilon}_1\bar{\varepsilon}_2 e_2 & R(z, e_2)e_2 &= \frac{1}{4}\varepsilon\bar{\varepsilon}_1 z \\
R(v, e_1)z &= -\frac{1}{2}\varepsilon\bar{\varepsilon}_2 u & R(v, e_1)v &= \frac{1}{2}\bar{\varepsilon}_2 z \\
R(e_1, e_2)e_1 &= \frac{3}{4}\varepsilon\bar{\varepsilon}_2 e_2 & R(e_1, e_2)e_2 &= -\frac{3}{4}\varepsilon\bar{\varepsilon}_1 e_1
\end{aligned}$$

Let $x, y \in \mathfrak{n}$. Recall that homaloidal planes are those for which the numerator $\langle R(x, y)y, x \rangle$ of the sectional curvature formula vanishes. This notion is useful for degenerate planes tangent to spaces that are not of constant curvature.

Theorem 3.11 *All central planes are homaloidal: $R(z, z')z'' = R(u, x)y = R(x, y)u = 0$ for all $z, z', z'' \in \mathfrak{Z}$, $u \in \mathfrak{U}$, and $x, y \in \mathfrak{n}$. Thus the nondegenerate part of the center is flat:*

$$K(z, z') = 0.$$

□

This recovers [11, (2.4), item c)] in the Riemannian case.

In view of this result, we shall extend the notion of flatness to possibly degenerate submanifolds.

Definition 3.12 *A submanifold of a pseudoriemannian manifold is flat if and only if every plane tangent to the submanifold is homaloidal.*

Corollary 3.13 *The center Z of N is flat.* □

Corollary 3.14 *The only N of constant curvature are flat.*

Proof: If $\dim Z > 1$, this follows immediately from the previous Corollary. If $\dim Z = 1$ and $\dim N \geq 4$, then from Example 4.2 *infra* there are abelian subgroups of dimension greater than or equal to 2 which give rise to flat submanifolds. Finally, the nonflat Heisenberg group has nonconstant curvature. □

The degenerate part of the center can have a profound effect on the geometry of the whole group.

Theorem 3.15 *If $[\mathfrak{n}, \mathfrak{n}] \subseteq \mathfrak{U}$ and $\mathfrak{E} = \{0\}$, then N is flat.*

Proof: To begin, observe that the first part of the hypothesis implies that $j(z) = 0$ for all $z \in \mathfrak{Z}$. This eliminates most of the possible contributions to a nonzero curvature operator. With the second part included, one need only check (3.17), and this is readily seen to vanish as well. □

Among these spaces, those that also have $\mathfrak{Z} = \{0\}$ (which condition itself implies $[\mathfrak{n}, \mathfrak{n}] \subseteq \mathfrak{U}$) are fundamental, with the more general ones obtained by making nondegenerate central extensions. It is also easy to see that the product of any flat group with a nondegenerate abelian factor is still flat.

This is the best possible result in general. Using weaker hypotheses in place of $\mathfrak{E} = \{0\}$, such as $[\mathfrak{V}, \mathfrak{V}] = \{0\} = [\mathfrak{E}, \mathfrak{E}]$, it is easy to construct examples which are not flat.

Corollary 3.16 *If $\dim Z \geq \lceil \frac{n}{2} \rceil$, then there exists a flat metric on N .* □

Here $\lceil r \rceil$ denotes the least integer greater than or equal to r .

Before continuing, we pause to collect some facts about the condition $[\mathfrak{n}, \mathfrak{n}] \subseteq \mathfrak{U}$ and its consequences.

Remarks 3.17 Since it implies $j(z) = 0$ for all $z \in \mathfrak{Z}$, this latter is possible with no pseudo-euclidean de Rham factor, unlike the Riemannian case.

Also, it implies $j(u)$ interchanges \mathfrak{V} and \mathfrak{E} for all $u \in \mathfrak{U}$ if and only if $[\mathfrak{V}, \mathfrak{V}] = [\mathfrak{E}, \mathfrak{E}] = \{0\}$. Thus an N of pH -type can have $[\mathfrak{n}, \mathfrak{n}] \subseteq \mathfrak{U}$ if and only if $\mathfrak{Z} = \{0\}$. Examples are the Heisenberg group and the groups $H(p, 1)$ for $p \geq 2$ with null centers.

Finally we note it implies that, for every $u \in \mathfrak{U}$, $j(u)$ maps \mathfrak{V} to \mathfrak{V} if and only if $j(u)$ maps \mathfrak{E} to \mathfrak{E} if and only if $[\mathfrak{V}, \mathfrak{E}] = \{0\}$.

We now continue with some general formulas for sectional curvature.

Theorem 3.18 *For any orthonormal $z \in \mathfrak{Z}$ and $e \in \mathfrak{E}$,*

$$K(z, e) = \frac{1}{4} \varepsilon_z \bar{\varepsilon}_e \langle j(\iota z)e, j(\iota z)e \rangle$$

with $\varepsilon_z = \langle z, z \rangle$ and $\bar{\varepsilon}_e = \langle e, e \rangle$. □

This recovers [11, (2.4), item b)] in the Riemannian case.

Theorem 3.19 *If e, e' are any orthonormal vectors in \mathfrak{E} , then*

$$K(e, e') = -\frac{3}{4} \bar{\varepsilon} \bar{\varepsilon}' \langle [e, e'], [e, e'] \rangle$$

with $\bar{\varepsilon} = \langle e, e \rangle$ and $\bar{\varepsilon}' = \langle e', e' \rangle$. □

This recovers [11, (2.4) item a)] in the Riemannian case.

Some other sectional curvature numerators are also relevant.

Proposition 3.20 *If $z \in \mathfrak{Z}$, $v \in \mathfrak{V}$, and $e \in \mathfrak{E}$, then*

$$\begin{aligned} \langle R(z, v)v, z \rangle &= \frac{1}{4} \langle j(\iota z)v, j(\iota z)v \rangle, \\ \langle R(v, e)e, v \rangle &= -\frac{3}{4} \langle [v, e], [v, e] \rangle + \frac{1}{4} \langle j(\iota v)e, j(\iota v)e \rangle, \\ \langle R(v, v')v', v \rangle &= -\frac{3}{4} \langle [v, v'], [v, v'] \rangle + \frac{1}{2} \langle j(\iota v)v', j(\iota v')v \rangle \\ &\quad + \frac{1}{4} \left(\langle j(\iota v')v, j(\iota v')v \rangle + \langle j(\iota v)v', j(\iota v)v' \rangle \right) \\ &\quad - \langle j(\iota v)v, j(\iota v')v' \rangle. \end{aligned}$$
□

Example 3.21 Continuing from Examples 2.4, 3.2, and 3.7, we find non-trivial sectional curvatures for the Heisenberg group with non-null center.

$$\begin{aligned} K(z, e_1) &= K(z, e_2) = \frac{1}{4} \varepsilon_1 \bar{\varepsilon}_2, \\ K(e_1, e_2) &= -\frac{3}{4} \varepsilon_1 \bar{\varepsilon}_2. \end{aligned}$$

Example 3.22 Continuing from Examples 2.6, 3.4, and 3.9, we find sectional curvatures for our quaternionic Heisenberg group.

$$\begin{aligned}\langle R(v_1, v_2)v_2, v_1 \rangle &= -(\bar{\varepsilon}_1 + \tfrac{3}{4}\varepsilon), \\ \langle R(v, e)e, v \rangle &= \langle R(z, v)v, z \rangle = 0, \\ K(z, e_1) &= K(z, e_2) = \tfrac{1}{4}\varepsilon\bar{\varepsilon}_1\bar{\varepsilon}_2, \\ K(e_1, e_2) &= -\tfrac{3}{4}\varepsilon\bar{\varepsilon}_1\bar{\varepsilon}_2.\end{aligned}$$

Thus ι cannot integrate to an isometry of N in general, as mentioned after equation (2.1). Isometries must preserve vanishing of sectional curvature, and an integral of ι would interchange homaloidal and nonhomaloidal planes in this example.

Example 3.23 Continuing from Examples 2.7, 3.5, and 3.10, we find sectional curvatures for the group $H(1, 2)$.

$$\begin{aligned}\langle R(v, e)e, v \rangle &= \langle R(z, v)v, z \rangle = 0, \\ K(z, e_1) &= K(z, e_2) = \tfrac{1}{4}\varepsilon\bar{\varepsilon}_1\bar{\varepsilon}_2, \\ K(e_1, e_2) &= -\tfrac{3}{4}\varepsilon\bar{\varepsilon}_1\bar{\varepsilon}_2.\end{aligned}$$

The Ricci curvature is a symmetric (0,2)-tensor given by

$$\text{Ric}(x, y) = \text{tr}(\xi \mapsto R(\xi, x)y).$$

With respect to the basis $\{u_i, z_\alpha, v_i, e_a\}$, we have

$$\text{Ric}(x, y) = \sum_i \langle R(v_i, x)y, u_i \rangle + \sum_\alpha \varepsilon_\alpha \langle R(z_\alpha, x)y, z_\alpha \rangle + \sum_a \bar{\varepsilon}_a \langle R(e_a, x)y, e_a \rangle$$

with $\varepsilon_\alpha = \langle z_\alpha, z_\alpha \rangle$ and $\bar{\varepsilon}_a = \langle e_a, e_a \rangle$, where we noted that terms of the form $\langle R(u_i, x)y, v_i \rangle = 0$ for all i .

Theorem 3.24 *Let N be a 2-step nilpotent Lie group with left-invariant pseudoriemannian metric tensor $\langle \cdot, \cdot \rangle$ and Lie algebra $\mathfrak{n} = \mathfrak{U} \oplus \mathfrak{Z} \oplus \mathfrak{V} \oplus \mathfrak{E}$ decomposed as in Section 2. For all $u \in \mathfrak{U}$, $z, z' \in \mathfrak{Z}$, $v, v' \in \mathfrak{V}$, $e \in \mathfrak{E}$, and $x \in \mathfrak{V} \oplus \mathfrak{E}$, we have*

$$\text{Ric}(u, \cdot) = 0, \tag{3.25}$$

$$\text{Ric}(z, z') = \tfrac{1}{4} \sum_a \bar{\varepsilon}_a \langle j(\iota z)e_a, j(\iota z')e_a \rangle, \tag{3.26}$$

$$\text{Ric}(v, z) = \frac{1}{4} \sum_a \bar{\varepsilon}_a \langle j(\iota v) e_a, j(\iota z) e_a \rangle, \quad (3.27)$$

$$\text{Ric}(e, z) = 0, \quad (3.28)$$

$$\text{Ric}(v, v') = -\frac{1}{2} \sum_\alpha \varepsilon_\alpha \langle j(z_\alpha) v, j(z_\alpha) v' \rangle + \frac{1}{4} \sum_a \bar{\varepsilon}_a \langle j(\iota v) e_a, j(\iota v') e_a \rangle, \quad (3.29)$$

$$\text{Ric}(x, e) = -\frac{1}{2} \sum_\alpha \varepsilon_\alpha \langle j(z_\alpha) x, j(z_\alpha) e \rangle. \quad (3.30)$$

Proof: This is mostly just straight-forward computation, but there are some details to be noted carefully. We give one case to illustrate this, in which we use the selfadjointness of ι and the ι -skewsymmetry of j extensively. From the definition,

$$\text{Ric}(x, e) = \sum_\alpha \varepsilon_\alpha \langle R(x, z_\alpha) z_\alpha, e \rangle + \sum_a \bar{\varepsilon}_a \langle R(x, e_a) e_a, e \rangle.$$

From Theorem 3.6 and those properties of ι and j ,

$$\begin{aligned} \langle R(x, z_\alpha) z_\alpha, e \rangle &= -\frac{1}{4} \langle \iota j(\iota z_\alpha) [\iota j(\iota z_\alpha) x]^\mathfrak{E}, e \rangle \\ &= \frac{1}{4} \langle j(z_\alpha) x, j(z_\alpha) e \rangle. \end{aligned}$$

The basic trick in all these cases is to expand in an orthonormal basis and write

$$j(\iota[e_a, x]^3) = \sum_\alpha \varepsilon_\alpha \langle \iota j(\iota z_\alpha) e_a, x \rangle j(\iota z_\alpha).$$

Then we get

$$\begin{aligned} \langle R(e_a, x) e_a, e \rangle &= \frac{3}{4} \sum_\alpha \langle \iota j(\iota z_\alpha) e_a, x \rangle \langle \iota j(\iota z_\alpha) e_a, e \rangle \\ &= \frac{3}{4} \sum_\alpha \varepsilon_\alpha \langle j(\iota z_\alpha) x, e_a \rangle \langle j(\iota z_\alpha) e, e_a \rangle \\ &= \frac{3}{4} \sum_\alpha \varepsilon_\alpha \langle j(z_\alpha) x, e_a \rangle \langle j(z_\alpha) e, e_a \rangle, \end{aligned}$$

and summing on a yields

$$\sum_a \bar{\varepsilon}_a \langle R(e_a, x) e_a, e \rangle = \frac{3}{4} \sum_\alpha \varepsilon_\alpha \langle j(z_\alpha) x, j(z_\alpha) e \rangle.$$

This results in the formula for $\text{Ric}(x, e)$, and the others are similar. \square

Corollary 3.25 *If $j(z) = 0$ for all $z \in \mathfrak{Z}$ and $j(u)$ interchanges \mathfrak{V} and \mathfrak{E} for all $u \in \mathfrak{U}$, then N is Ricci flat.* \square

Theorem 3.26 *Let N be a 2-step nilpotent Lie group with left-invariant pseudoriemannian metric tensor $\langle \cdot, \cdot \rangle$ and Lie algebra $\mathfrak{n} = \mathfrak{U} \oplus \mathfrak{Z} \oplus \mathfrak{V} \oplus \mathfrak{E}$ decomposed as in Section 2. The scalar curvature is given by*

$$S = -\frac{1}{4} \sum_{\alpha, a} \varepsilon_\alpha \bar{\varepsilon}_a \langle j(z_\alpha) e_a, j(z_\alpha) e_a \rangle. \quad \square$$

Corollary 3.27 *If $j(z) = 0$ for all $z \in \mathfrak{Z}$, then N is scalar flat. In particular, this occurs when $[\mathfrak{n}, \mathfrak{n}] \subseteq \mathfrak{U}$.*

Proof: The second part follows from the observation at the beginning of the proof of Theorem 3.15. \square

Example 3.28 Continuing from Examples 2.4, 3.2, 3.7, and 3.21, we find nontrivial Ricci curvatures for the Heisenberg group with non-null center.

$$\begin{aligned} \text{Ric}(z, z) &= \frac{1}{2} \varepsilon_1 \bar{\varepsilon}_2 \\ \text{Ric}(e_1, e_1) &= -\frac{1}{2} \varepsilon \bar{\varepsilon}_2 \\ \text{Ric}(e_2, e_2) &= -\frac{1}{2} \varepsilon \bar{\varepsilon}_1 \end{aligned}$$

The scalar curvature is $S = -\frac{1}{2} \varepsilon \bar{\varepsilon}_1 \bar{\varepsilon}_2$.

For the group $H(p, 1)$ with non-null center, we take a basis of the Lie algebra $\{z, e_1, \dots, e_p, e_{p+1}, \dots, e_{2p}\}$ with structure equation $[e_i, e_{p+i}] = z$ for $1 \leq i \leq p$. Then we find

$$\begin{aligned} \text{Ric}(z, z) &= \frac{1}{2} \sum_{i=1}^p \bar{\varepsilon}_i \bar{\varepsilon}_{p+i}, \\ \text{Ric}(e_i, e_i) &= -\frac{1}{2} \varepsilon \bar{\varepsilon}_{p+i}, \\ \text{Ric}(e_{p+i}, e_{p+i}) &= -\frac{1}{2} \varepsilon \bar{\varepsilon}_i. \end{aligned}$$

The scalar curvature is $S = -\frac{1}{2} \varepsilon \sum_i \bar{\varepsilon}_i \bar{\varepsilon}_{p+i}$.

Example 3.29 Continuing from Examples 2.5, 3.3, and 3.8, since the Heisenberg group with null center is flat, it is also Ricci flat and scalar flat.

For the group $H(p, 1)$ with $p \geq 2$ and null center, we take a basis of the Lie algebra $\{u, v, e_2, \dots, e_p, e_{p+1}, \dots, e_{2p}\}$ with structure equations $[v, e_{p+1}] = u$ and $[e_i, e_{p+i}] = u$ for $2 \leq i \leq p$. Then we find

$$\text{Ric}(v, v) = \frac{1}{2} \sum_{i=2}^p \bar{\varepsilon}_i \bar{\varepsilon}_{p+i},$$

but the scalar curvature vanishes: $S = 0$. Note that for p odd, with an appropriate choice of signature one obtains examples of metrics that are not flat, but are Ricci and scalar flat.

Example 3.30 Continuing from Examples 2.6, 3.4, 3.9, and 3.22, we find nontrivial Ricci curvatures for our quaternionic Heisenberg algebra.

$$\begin{aligned}\mathrm{Ric}(z, z) &= \frac{1}{2}\bar{\varepsilon}_1\bar{\varepsilon}_2 \\ \mathrm{Ric}(e_1, e_1) &= -\frac{1}{2}\varepsilon\bar{\varepsilon}_2 \\ \mathrm{Ric}(e_2, e_2) &= -\frac{1}{2}\varepsilon\bar{\varepsilon}_1\end{aligned}$$

The scalar curvature is $S = -\frac{1}{2}\varepsilon\bar{\varepsilon}_1\bar{\varepsilon}_2$.

Example 3.31 Continuing from Examples 2.7, 3.5, 3.10, and 3.23, we find nontrivial Ricci curvatures for the group $H(1, 2)$.

$$\begin{aligned}\mathrm{Ric}(z, z) &= \frac{1}{2}\bar{\varepsilon}_1\bar{\varepsilon}_2 \\ \mathrm{Ric}(e_1, e_1) &= -\frac{1}{2}\varepsilon\bar{\varepsilon}_2 \\ \mathrm{Ric}(e_2, e_2) &= -\frac{1}{2}\varepsilon\bar{\varepsilon}_1\end{aligned}$$

The scalar curvature is $S = -\frac{1}{2}\varepsilon\bar{\varepsilon}_1\bar{\varepsilon}_2$.

For the group $H(1, p)$ with $p \geq 2$, we take a basis $\{u_1, \dots, u_q, z_1, \dots, z_{p-q}, v_1, \dots, v_q, e_1, \dots, e_{p-q}, e\}$ with structure equations $[v_i, e] = u_i$ for $1 \leq i \leq q$ and $[e_j, e] = z_j$ for $1 \leq j \leq p - q$. Then we find

$$\begin{aligned}\mathrm{Ric}(z_j, z_j) &= \frac{1}{2}\bar{\varepsilon}_j\bar{\varepsilon}, \\ \mathrm{Ric}(e_j, e_j) &= -\frac{1}{2}\varepsilon_j\bar{\varepsilon}, \\ \mathrm{Ric}(e, e) &= -\frac{1}{2}\sum_{j=1}^{p-q}\varepsilon_j\bar{\varepsilon}_j.\end{aligned}$$

The scalar curvature is $S = -\frac{1}{2}\bar{\varepsilon}\sum_j\varepsilon_j\bar{\varepsilon}_j$.

We note the amusing low-dimensional coincidence of three of these examples.

When N has a pseudoeuclidean de Rham factor, it may be characterized in terms of j ; cf. [11, (2.7)]. To use the pseudoriemannian de Rham theorem of Wu [26], we must assume that \mathfrak{z} is nondegenerate.

Proposition 3.32 *Let N be a simply connected, 2-step nilpotent Lie group with left-invariant metric tensor \langle, \rangle and nondegenerate center. Assume that N can be written as a pseudoriemannian product $E \times N'$ with E the pseudoeuclidean de Rham factor and N' the product of all other de Rham factors, and write $\mathfrak{n} = \mathfrak{e} \oplus \mathfrak{n}'$ as the corresponding orthogonal direct sum. Then $\mathfrak{e} \subseteq \mathfrak{z}$ and $\mathfrak{e} = \{z \in \mathfrak{z} \mid j(z) = j(\iota z) = 0\} = \ker j \cap \ker j\iota$.*

Proof: For convenience, we shall also denote the corresponding subbundles of TN by \mathfrak{e} and \mathfrak{n}' . Then we note that \mathfrak{e} is parallel, integrable, and ι -invariant. Also, $\mathfrak{n} = \mathfrak{z} \oplus \mathfrak{v}$ is an orthogonal direct sum and \mathfrak{z} and \mathfrak{v} are ι -invariant, since we have assumed \mathfrak{z} is nondegenerate.

First we show that $\mathfrak{e} \cap \mathfrak{v} = \{0\}$. Suppose not, with $0 \neq x \in \mathfrak{e} \cap \mathfrak{v}$. Then for all $z \in \mathfrak{z}$, we have $j(z)x = -2\iota\nabla_{\iota z}x$. Since \mathfrak{e} and \mathfrak{v} are ι -invariant, $\nabla_{\iota z}x \in \mathfrak{v}$, and \mathfrak{e} is parallel, we deduce that $j(z)x \in \mathfrak{e} \cap \mathfrak{v}$ for all $z \in \mathfrak{z}$. To see that $j(z)x \neq 0$ for some $z \in \mathfrak{z}$, choose $y \in \mathfrak{v}$ such that $[x, y] \neq 0$. Then for $z = [x, y]$, we find $\langle \iota j(z)x, y \rangle = \langle \text{ad}_x^\dagger \iota z, y \rangle = \langle [x, y], \iota z \rangle = \langle z, \iota z \rangle \neq 0$. Thus $\iota j(z)x \neq 0$ whence

$$0 \neq x \in \mathfrak{v} \text{ implies } j(z)x \neq 0 \text{ for some } z \in \mathfrak{z}. \quad (3.31)$$

Now, for any such z , we have $[x, j(z)x] \in \mathfrak{e} \cap \mathfrak{z}$ because $x \in \mathfrak{e} \cap \mathfrak{v}$, $j(z)x \in \mathfrak{e} \cap \mathfrak{v}$, and \mathfrak{e} is integrable. Moreover, for such a z , we would have $\langle [x, j(z)x], \iota z \rangle = \langle \iota j(z)x, j(z)x \rangle \neq 0$ and $[x, j(z)x]$ would be another nonzero element in $\mathfrak{e} \cap \mathfrak{v}$. But this would mean that a sectional curvature numerator for $\llbracket x, j(z)x \rrbracket$, namely $\langle [x, j(z)x], [x, j(z)x] \rangle$, is nonvanishing. This contradicts E being flat, so we conclude that $\mathfrak{e} \cap \mathfrak{v} = \{0\}$ as desired.

Next we show that $\mathfrak{e} \subseteq \mathfrak{z}$. Let $0 \neq a \in \mathfrak{e}$ and write $a = z + x$ with $z \in \mathfrak{z}$ and $x \in \mathfrak{v}$. To show that $x = 0$ it suffices by (3.31) to show that $j(z')x = 0$ for all $z' \in \mathfrak{z}$. Let $z' \in \mathfrak{z}$ be fixed but arbitrary. Then the parallel and ι -invariant distribution \mathfrak{e} contains $\nabla_{\iota z'}a = \nabla_{\iota z'}x = -\frac{1}{2}\iota j(z')x$. But from the first part of this proof it follows that $j(z')x = 0$ since $j(z')x \in \mathfrak{e} \cap \mathfrak{v}$. Therefore $\mathfrak{e} \subseteq \mathfrak{z}$.

We now claim that for all $z \in \mathfrak{e}$, we have $j(z) = j(\iota z) = 0 \in \text{End}(\mathfrak{v})$. Indeed, for any $z \in \mathfrak{e}$ and $x \in \mathfrak{v}$, since \mathfrak{e} is parallel and ι -invariant, $\iota\nabla_x z = -\frac{1}{2}j(\iota z)x$. But $j(\iota z)x \in \mathfrak{e} \cap \mathfrak{v} = \{0\}$; similarly, $j(z)x = 0$.

Finally, we show that if $j(z) = j(\iota z) = 0$ then $z \in \mathfrak{e}$. For any such z set $\mathfrak{e}' = \llbracket \mathfrak{e}, z \rrbracket$, and observe that it suffices to prove that \mathfrak{e}' is parallel and flat. Consider $a' \in \mathfrak{e}'$ and $x' \in \mathfrak{n}$ and write $a' = a + tz$ for some $a \in \mathfrak{e}$ and $t \in \mathbb{R}$, and $x' = z' + x$ for some $z' \in \mathfrak{z}$ and $x \in \mathfrak{v}$. Then

$$\nabla_{x'}a' = \nabla_{x'}a + t\nabla_{x'}z = \nabla_{x'}a - \frac{t}{2}\iota j(\iota z)x = \nabla_{x'}a \in \mathfrak{e} \subseteq \mathfrak{e}'$$

so \mathfrak{e}' is parallel. Now, from $\mathfrak{e}' \subseteq \mathfrak{z}$ it follows that $\nabla_e e' = 0$ for all $e, e' \in \mathfrak{e}'$, whence \mathfrak{e}' is flat. \square

4 Totally Geodesic Subgroups and Geodesics

We begin by noting that O'Neill [20, Ex. 9, p. 125] has extended the definition of totally geodesic to degenerate submanifolds of pseudoriemannian manifolds. We shall use this extended version.

Recall from [20] that the extrinsic and intrinsic curvatures of totally geodesic submanifolds coincide. Thus there is an unambiguous notion of flatness for them. Note that a connected subgroup N' of N is a totally geodesic submanifold if and only if it is totally geodesic at the identity element of N , because left translations by elements of N' are isometries of N that leave N' invariant. A connected, totally geodesic submanifold need not be a connected, totally geodesic subgroup, but (N, \langle, \rangle) has many totally geodesic subgroups. Many of them are flat, illustrating the similarity to pseudoeuclidean spaces; cf. [11, (2.11)].

Example 4.1 For any $x \in \mathfrak{n}$ the 1-parameter subgroup $\exp(tx)$ is a geodesic if and only if $\text{ad}_x^\dagger x = 0$. We find this if and only if $x \in \mathfrak{z}$ or $x \in \mathfrak{U} \oplus \mathfrak{E}$. This is essentially the same as the Riemannian case, but with some additional geodesic 1-parameter subgroups coming from \mathfrak{U} .

Example 4.2 Abelian subspaces of $\mathfrak{V} \oplus \mathfrak{E}$ are Lie subalgebras of \mathfrak{n} , and give rise to complete, flat, totally geodesic abelian subgroups of N , just as in the Riemannian case [11]. The construction given in [11, (2.11), Ex. 2] is valid in general, and shows that if $\dim \mathfrak{V} \oplus \mathfrak{E} \geq 1 + k + k \dim \mathfrak{z}$, then every nonzero element of $\mathfrak{V} \oplus \mathfrak{E}$ lies in an abelian subspace of dimension $k + 1$.

Example 4.3 The center Z of N is a complete, flat, totally geodesic submanifold. Moreover, it determines a foliation of N by its left translates, so each leaf is flat and totally geodesic, as in the Riemannian case [11]. In the pseudoriemannian case, this foliation in turn is the orthogonal direct sum of two foliations determined by \mathfrak{U} and \mathfrak{z} , and the leaves of the \mathfrak{U} -foliation are also null. All these leaves are complete.

Let γ be a geodesic in a group N of pH -type. Without loss of generality, we may assume that $\gamma(0) = 1 \in N$. Write $\dot{\gamma}(0) = a_0 + x_0 \in (\mathfrak{U} \oplus \mathfrak{z}) \oplus (\mathfrak{V} \oplus \mathfrak{E})$ and suppose that neither a_0 nor x_0 is zero. Consider the span $\llbracket a_0, x_0, j(a_0)x_0 \rrbracket = \mathfrak{n}'$ in the Lie algebra \mathfrak{n} of N . The following lemma shows that \mathfrak{n}' is isomorphic to the 3-dimensional Heisenberg algebra \mathfrak{h}_3 .

Lemma 4.4 *Let $\mathfrak{n} = \mathfrak{U} \oplus \mathfrak{z} \oplus \mathfrak{V} \oplus \mathfrak{E}$ decomposed as in Section 2 be an algebra of pH -type. If $x \in \mathfrak{V} \oplus \mathfrak{E}$ and $a \in \mathfrak{U} \oplus \mathfrak{z}$, then $[x, j(a)x] = \langle x, \iota x \rangle a$.*

Proof: For any $a' \in \mathfrak{U} \oplus \mathfrak{Z}$, using (2.3) and the third identity from Proposition 2.9, we have

$$\begin{aligned}\langle [x, j(a)x], \iota a' \rangle &= \langle j(a)x, \iota j(a')x \rangle \\ &= \langle x, \iota x \rangle \langle a, \iota a' \rangle.\end{aligned}$$

Thus $\langle a, \iota a' \rangle = 0$ implies there is no component in the a' direction. \square

In the Riemannian case, each such subgroup N' with Lie algebra \mathfrak{n}' is totally geodesic [11, p. 624f]. But in general, this is not true.

Example 4.5 In our quaternionic Heisenberg group from Example 2.6, consider a geodesic γ with $\dot{\gamma}(0) = a_0 + x_0 = (u_2 + z) + (v_1 + e_1)$. Then one computes easily $j(a_0)x_0 = 2v_2$, so from Example 3.4 we find $\nabla_{x_0} a_0 = -\frac{1}{2}\varepsilon(u_2 + \bar{\varepsilon}_2 e_2) \notin \mathfrak{n}' = \llbracket u_2 + z, v_1 + e_1, v_2 \rrbracket$.

As usual, part of the problem is the presence of the null subspaces \mathfrak{U} and \mathfrak{V} . But even if we assume that \mathfrak{Z} is nondegenerate, these subgroups still need not be totally geodesic.

Example 4.6 For the group $H(p, 1)$ with $p \geq 2$, take $a = z$ and $x = e_i + e_j$ with $i \neq j$. Then $j(a)x = j(z)(e_i + e_j) = e_{p+i} + e_{p+j}$ and

$$\nabla_a x = -\frac{1}{2}\varepsilon(\bar{\varepsilon}_{p+i}e_{p+i} + \bar{\varepsilon}_{p+j}e_{p+j}) \notin \llbracket a, x, j(a)x \rrbracket.$$

One may also check that using, for example, $\iota j(\iota a)x = \text{ad}_x^\dagger a$ will not help: in the quaternionic Heisenberg algebra with $a = z_1$ and $x = e_1 + e_4$, $\llbracket a, x, \text{ad}_x^\dagger a \rrbracket$ is totally geodesic but is not a subalgebra.

For the geodesic equation, let us consider (as suffices) a geodesic γ with $\gamma(0) = 1 \in N$ and $\dot{\gamma}(0) = a_0 + x_0 \in \mathfrak{Z} \oplus \mathfrak{v}$. Further decompose $a_0 = u_0 + z_0 \in \mathfrak{U} \oplus \mathfrak{Z}$ and $x_0 = v_0 + e_0 \in \mathfrak{V} \oplus \mathfrak{E}$. In exponential coordinates, write $\gamma(t) = \exp(a(t) + x(t)) = \exp(u(t) + z(t) + v(t) + e(t))$ with $\dot{a}(0) = a_0$, $\dot{x}(0) = x_0$, etc. For the tangent vector $\dot{\gamma}$, we obtain

$$\begin{aligned}\dot{\gamma} &= \exp_{(a+x)*}(\dot{a} + \dot{x}) \\ &= L_{\gamma(t)*}(\dot{a} + \dot{x} + \tfrac{1}{2}[\dot{a} + \dot{x}, a + x]) \\ &= L_{\gamma(t)*}(\dot{a} + \dot{x} + \tfrac{1}{2}[\dot{x}, x]),\end{aligned}$$

using Lemma 1.1, regarded as vector fields along γ . Then the geodesic equation is equivalent to

$$\frac{d}{dt}(\dot{a} + \dot{x} + \tfrac{1}{2}[\dot{x}, x]) - \text{ad}_{\dot{a} + \dot{x} + \frac{1}{2}[\dot{x}, x]}^\dagger(\dot{a} + \dot{x} + \tfrac{1}{2}[\dot{x}, x]) = 0.$$

Simplifying slightly, we find

$$\underbrace{\frac{d}{dt}(\dot{a} + \frac{1}{2}[\dot{x}, x])}_{\in \mathfrak{U} \oplus \mathfrak{Z}} + \underbrace{\ddot{x}}_{\in \mathfrak{V} \oplus \mathfrak{E}} - \underbrace{\text{ad}_x^\dagger(\dot{z} + \dot{v} + \frac{1}{2}[\dot{x}, x])}_{\in \mathfrak{U} \oplus \mathfrak{E}} = 0. \quad (4.1)$$

Using superscripts to denote components as before, we obtain for the \mathfrak{Z} -component

$$\frac{d}{dt}(\dot{z} + \frac{1}{2}[\dot{x}, x]^{\mathfrak{Z}}) = 0.$$

Using the initial condition, we get

$$\dot{z} + \frac{1}{2}[\dot{x}, x]^{\mathfrak{Z}} = z_0.$$

Next we note that $\ddot{v} = 0$ whence $\dot{v}(t) = v_0$ is a constant. We use these to simplify the other component equations.

$$\frac{d}{dt}(\dot{u} + \frac{1}{2}[\dot{x}, x]^{\mathfrak{U}}) - (\text{ad}_x^\dagger(z_0 + v_0))^{\mathfrak{U}} = 0 \quad (4.2)$$

$$\dot{z} + \frac{1}{2}[\dot{x}, x]^{\mathfrak{Z}} = z_0 \quad (4.3)$$

$$\ddot{v} = 0 \quad (4.4)$$

$$\ddot{e} - (\text{ad}_x^\dagger(z_0 + v_0))^{\mathfrak{E}} = 0 \quad (4.5)$$

In analogy with Eberlein [10, 11] we define two operators.

Definition 4.7 For fixed $z_0 \in \mathfrak{Z}$ and $v_0 \in \mathfrak{V}$ as above, define

$$\begin{aligned} \mathcal{J} : \mathfrak{V} \oplus \mathfrak{E} &\longrightarrow \mathfrak{E} : y \longmapsto (\text{ad}_y^\dagger(z_0 + v_0))^{\mathfrak{E}}, \\ \mathcal{J} : \mathfrak{V} \oplus \mathfrak{E} &\longrightarrow \mathfrak{U} : y \longmapsto (\text{ad}_y^\dagger(z_0 + v_0))^{\mathfrak{U}}. \end{aligned}$$

We shall denote the restriction of \mathcal{J} to \mathfrak{E} by J , and this will play the same role as J in Eberlein [10, 11].

Now we rewrite the geodesic equations in terms of J and \mathcal{J} , using the linearity of ad^\dagger to rearrange some terms.

$$\frac{d}{dt}(\dot{u} + \frac{1}{2}[\dot{x}, x]^{\mathfrak{U}}) = \mathcal{J}\dot{x} \quad (4.6)$$

$$\dot{z} + \frac{1}{2}[\dot{x}, x]^{\mathfrak{Z}} = z_0 \quad (4.7)$$

$$\ddot{v} = 0 \quad (4.8)$$

$$\ddot{e} - J\dot{e} = \mathcal{J}v_0 \quad (4.9)$$

While the \mathfrak{V} -component of a geodesic is simple, its mere presence affects all of the other components.

We also readily see that the system is completely integrable. Thus, as noted in the Introduction (proof of Theorem 1.5), all left-invariant pseudoriemannian metrics on these groups are complete. Also, *regardless of signature*, we may obtain the existence of $\dim \mathfrak{z}$ first integrals as in [11].

Keep $J \in \text{End}(\mathfrak{E})$ and write $\mathfrak{E} = \mathfrak{E}_1 \oplus \mathfrak{E}_2$ with $\mathfrak{E}_1 = \ker J$, an orthogonal direct sum. Decompose $\mathcal{J}v_0 = y_1 + y_2 \in \mathfrak{E}_1 \oplus \mathfrak{E}_2$, respectively. Note that J is invertible on \mathfrak{E}_2 ; we denote this restriction by J also. Now we follow Eberlein [11, (3.2–3.4)] for these next three results.

Proposition 4.8 *Let N be simply connected, γ a geodesic with $\gamma(0) = 1 \in N$, and $\dot{\gamma}(0) = a_0 + x_0 \in \mathfrak{n}$. Then*

$$\dot{\gamma} = L_{\gamma(t)*} (a_0 + e^{tJ}(e_0 + J^{-1}y_2) + v_0 - J^{-1}y_2 + t y_1 + \mathcal{J}\mathcal{J}\dot{x})$$

where $\mathcal{J}\mathcal{J}\dot{x} = \int_0^t \mathcal{J}\dot{x}(s) ds$. □

Corollary 4.9 *Let Φ_t denote the geodesic flow in TN , $n \in N$, and $a_0 + x_0 \in \mathfrak{n}$. Then*

$$\Phi_t L_{n*}(a_0 + x_0) = L_{\gamma(t)*} (a_0 + e^{tJ}(e_0 + J^{-1}y_2) + v_0 - J^{-1}y_2 + t y_1 + \mathcal{J}\mathcal{J}\dot{x})$$

where γ is the unique geodesic with $\gamma(0) = n$ and $\dot{\gamma}(0) = L_{n*}(a_0 + x_0)$. □

The proofs are direct computations which we omit; see [10, 11].

In the next result, we use PTM to denote the projectivized tangent bundle of a manifold M .

Corollary 4.10 *Let $\{z_1, \dots, z_r\}$ be any basis of \mathfrak{z} and let $\{z_1^*, \dots, z_r^*\}$ be the dual basis of \mathfrak{z}^* . Define $f_0 : TN \rightarrow \mathfrak{z} : L_{n*}x \mapsto \pi x$ where $n \in N$, $x \in \mathfrak{n}$, and $\pi : \mathfrak{n} \rightarrow \mathfrak{z}$ is the projection. For $1 \leq \alpha \leq r$, define $f_\alpha : TN \rightarrow \mathbb{R}$ by*

$$f_\alpha = z_\alpha^* \circ f_0, \quad 1 \leq \alpha \leq r.$$

Then for $0 \leq \alpha \leq r$, $t \in \mathbb{R}$, and $n \in N$, we have these:

1. $f_\alpha \circ \Phi_t = f_\alpha$.
2. $f_\alpha \circ L_{n*} = f_\alpha$.

3. If $\Gamma \leq N$ is any discrete subgroup acting on N by left translations, then f_α induces $F_\alpha : T(\Gamma \backslash N) \rightarrow \mathbb{R}$ with $F_\alpha \circ \Phi_t = F_\alpha$ for Φ_t the induced geodesic flow on $T(\Gamma \backslash N)$. In particular, Φ_t has no dense orbit in any of the three parts (timelike, spacelike, or null) of $PT(\Gamma \backslash N)$.
4. The functions f_α and F_α for $1 \leq \alpha \leq r$ are linearly independent over \mathbb{R} .

Proof: The first two parts are immediate consequences of Corollary 4.9. Letting $p_N : N \rightarrow \Gamma \backslash N$ denote the natural projection, the second part implies that the functions $F_0 : T(\Gamma \backslash N) \rightarrow \mathfrak{z}$ and $F_\alpha : T(\Gamma \backslash N) \rightarrow \mathbb{R}$ for $\alpha \geq 1$ are well defined and satisfy $F_\alpha p_{N*} = f_\alpha$ for all α . From the first part, the F_α commute with Φ_t . The functions F_α are clearly continuous and nonconstant in $PT(\Gamma \backslash N)$, but constant along Φ_t -orbits. Thus there is no dense orbit in any part of $PT(\Gamma \backslash N)$. Finally, the last part follows from the definition of F_α and f_α . \square

We continue with a geodesic through the identity element, so $\gamma(0) = 1$ and $\dot{\gamma}(0) = a_0 + x_0 = u_0 + z_0 + v_0 + e_0$, with $J \in \text{End}(\mathfrak{E})$ and $\mathfrak{E} = \mathfrak{E}_1 \oplus \mathfrak{E}_2$ with $\mathfrak{E}_1 = \ker J$, and with $\mathcal{J}v_0 = y_1 + y_2 \in \mathfrak{E}_1 \oplus \mathfrak{E}_2$ as before. Decompose $e_0 = e_1 + e_2 \in \mathfrak{E}_1 \oplus \mathfrak{E}_2$, respectively. (These e_i should not be confused with the basis elements appearing in other sections.) Observe that J is skewadjoint with respect to $\langle \cdot, \cdot \rangle$. Recall that J is invertible on \mathfrak{E}_2 , and that we let J denote the restriction there as well. For convenience, set

$$\begin{aligned} x_1 &= e_1 + v_0 - J^{-1}y_2, \\ x_2 &= e_2 + J^{-1}y_2. \end{aligned}$$

Theorem 4.11 *Using these notations, the geodesic equations may be integrated as:*

$$x(t) = t x_1 + (e^{tJ} - I) J^{-1} x_2 + \frac{1}{2} t^2 y_1, \quad (4.10)$$

$$z(t) = t z_0 + \mathcal{J}[\dot{x}, x]^3, \quad (4.11)$$

$$u(t) = t u_0 + \mathcal{J}[\dot{x}, x]^4 + \mathcal{J} \mathcal{J} \dot{x}, \quad (4.12)$$

where

$$\begin{aligned} \mathcal{J}[\dot{x}, x] &= -\frac{1}{2} \int_0^t [\dot{x}(s), x(s)] ds \\ &= \frac{1}{2} t [x_1 + \frac{1}{2} t y_1, (e^{tJ} + I) J^{-1} x_2] - t [y_1, e^{tJ} x_2] + \frac{1}{12} t^3 [x_1, y_1] \end{aligned}$$

$$\begin{aligned}
& + \frac{1}{2} [(e^{tJ} - I) J^{-1} x_2, J^{-1} x_2] - [x_1, (e^{tJ} - I) J^{-2} x_2] \\
& + [y_1, (e^{tJ} - I) J^{-3} x_2] + \frac{1}{2} \int_0^t [e^{sJ} J^{-1} x_2, e^{sJ} x_2] ds
\end{aligned}$$

and

$$\mathcal{L} \mathcal{J} \dot{x} = \int_0^t \int_0^s \mathcal{J} \dot{x}(\sigma) d\sigma ds.$$

Proof: The formulas follow from straightforward integrations of the geodesic equations (4.6)–(4.9). We used the general fact about exponentials of matrices that J commutes with e^{tJ} for all $t \in \mathbb{R}$. Using this, it is routine to verify that $x(t)$, $z(t)$, and $u(t)$ satisfy the geodesic equations and initial conditions. \square

Corollary 4.12 *When N has a nondegenerate center, the formulas simplify somewhat. Now equation (4.9) is homogeneous and we obtain*

$$e(t) = t e_1 + (e^{tJ} - I) J^{-1} e_2, \quad (4.13)$$

$$z(t) = t z_1(t) + z_2(t) + z_3(t), \quad (4.14)$$

where

$$\begin{aligned}
z_1(t) &= z_0 + \frac{1}{2} [e_1, (e^{tJ} + I) J^{-1} e_2], \\
z_2(t) &= [e_1, (I - e^{tJ}) J^{-2} e_2] + \frac{1}{2} [e^{tJ} J^{-1} e_2, J^{-1} e_2], \\
z_3(t) &= \frac{1}{2} \int_0^t [e^{sJ} J^{-1} e_2, e^{sJ} e_2] ds.
\end{aligned}$$

Note that z_3 may contribute to z_1 and z_2 . \square

The flat spaces we found in Theorem 3.15 allow more simplification.

Corollary 4.13 *When $[\mathfrak{n}, \mathfrak{n}] \subseteq \mathfrak{U}$ and $\mathfrak{E} = \{0\}$, then $x = v$ and we obtain*

$$v(t) = t v_0, \quad (4.15)$$

$$z(t) = t z_0, \quad (4.16)$$

$$u(t) = t u_0 + \frac{1}{2} t^2 \mathcal{J} v_0. \quad (4.17)$$

\square

Corollary 4.14 *If $[\mathfrak{n}, \mathfrak{n}] \subseteq \mathfrak{U}$ and $\mathfrak{E} = \{0\}$, then N is geodesically connected. Consequently, so is any nilmanifold with such a universal covering space.*

Proof: N is complete by Theorem 1.5, hence pseudoconvex. The preceding geodesic equations show that N is nonreturning. Thus the space of geodesics $G(N)$ is Hausdorff by Theorem 5.2 of [4]. Now Theorem 4.2 of [3] yields geodesic connectedness of N . \square

Thus these compact nilmanifolds are much like tori. This is also illustrated by the computation of their period spectrum in Theorem 6.25.

When J^2 is diagonalizable, then the nondegenerate case also can be made completely explicit. For example, this happens when $\langle \cdot, \cdot \rangle$ restricted to \mathfrak{E}_2 is definite (*e.g.*, in the Riemannian case). The extreme instance is when J^2 is a scalar multiple of the identity. In the Riemannian case, this is much the same as H -type; however, in general this condition is not related to pH -type.

Let $\{\theta_1, \dots, \theta_k\}$ be the set of distinct nonzero eigenvalues of J^2 . Decompose \mathfrak{E}_2 as the orthogonal direct sum $\bigoplus_{j=1}^k \mathfrak{w}_j$ where J leaves each \mathfrak{w}_j invariant and $J^2|_{\mathfrak{w}_j} = \theta_j I$. Write $e_0 = e_1 + e_2$ with $e_i \in \mathfrak{E}_i$ and then write $e_2 = \sum_{j=1}^k w_j$ with each $w_j \in \mathfrak{w}_j$.

Proposition 4.15 *If N has a nondegenerate center and J^2 diagonalizes, then*

$$e(t) = t e_1 + (e^{tJ} - I) J^{-1} e_2, \quad (4.18)$$

$$z(t) = t z_1(t) + z_2(t), \quad (4.19)$$

where

$$\begin{aligned} z_1(t) &= z_0 + \frac{1}{2} [e_1, (e^{tJ} + I) J^{-1} e_2] + \frac{1}{2} \sum_{j=1}^k [J^{-1} w_j, w_j], \\ z_2(t) &= [e_1, (I - e^{tJ}) (J^{-2} e_2)] + \frac{1}{2} [e^{tJ} J^{-1} e_2, J^{-1} e_2] \\ &\quad - \frac{1}{2} \sum_{i \neq j} \frac{1}{\theta_j - \theta_i} ([e^{tJ} J w_i, e^{tJ} J^{-1} w_j] - [e^{tJ} w_i, e^{tJ} w_j]) \\ &\quad + \frac{1}{2} \sum_{i \neq j} \frac{1}{\theta_j - \theta_i} ([J w_i, J^{-1} w_j] - [w_i, w_j]). \end{aligned}$$

Proof: Compare Eberlein [11, (3.5)]. First note that on each \mathfrak{w}_j where the restriction J is invertible, $J = \theta_j J^{-1}$.

Next, one verifies straightforwardly that

$$[\dot{e}(t), e(t)] = [e_1, (e^{tJ} - I) J^{-1} e_2] + t [e^{tJ} e_2, e_1] + [e^{tJ} e_2, (e^{tJ} - I) J^{-1} e_2], \quad (4.20)$$

and, using the properties of J , that the derivative with respect to t of each $[e^{tJ}w_j, e^{tJ}J^{-1}w_j]$ is zero. Hence

$$[e^{tJ}w_j, e^{tJ}J^{-1}w_j] = [w_j, J^{-1}w_j]$$

for all t and j . From this and the decomposition of e_2 , we obtain

$$-2\dot{z}_3 = [e^{tJ}e_2, e^{tJ}J^{-1}e_2] = \sum_{i \neq j} [e^{tJ}w_i, e^{tJ}J^{-1}w_j] + \sum_j [w_j, J^{-1}w_j]. \quad (4.21)$$

Using the two facts about J again, another computation produces

$$\dot{z}_2 = -[e_1, e^{tJ}J^{-1}e_2] + \frac{1}{2}[e^{tJ}e_2, J^{-1}e_2].$$

Combining this with (4.21) and $\dot{z}(t) = z_1(t) + t\dot{z}_1 + \dot{z}_2 + \dot{z}_3$, we get

$$\begin{aligned} \dot{z}(t) = & z_0 - \frac{1}{2}[e^{tJ}e_2, e^{tJ}J^{-1}e_2] - \frac{1}{2}[e_1, (e^{tJ} - I)J^{-1}e_2] \\ & + \frac{1}{2}[e^{tJ}e_2, J^{-1}e_2] - \frac{t}{2}[e^{tJ}e_2, e_1]. \end{aligned}$$

Finally, it follows from (4.20) and this that z satisfies $\dot{z} + \frac{1}{2}[\dot{e}, e] = z_0$, and from (4.19) and this that z satisfies the initial conditions. \square

Next we recover [11, (3.7)] in our case.

Corollary 4.16 *Under the same hypotheses, $e(t) = te_1 + \sum_j \xi_j(t)$ where*

$$\xi_j(t) = \begin{cases} (\cos t\lambda_j - 1)J^{-1}w_j + \frac{\sin t\lambda_j}{\lambda_j}w_j & \text{for } \theta_j = -\lambda_j^2, \\ (\cosh t\lambda_j - 1)J^{-1}w_j + \frac{\sinh t\lambda_j}{\lambda_j}w_j & \text{for } \theta_j = \lambda_j^2, \end{cases}$$

with $\lambda_j > 0$.

Proof: First use the decomposition of e_2 . From the representation of J on \mathfrak{w}_j , we obtain in the first case

$$e^{tJ} = \cos t\lambda_j I + \frac{\sin t\lambda_j}{\lambda_j}J$$

there, and the second case is similar. \square

Thus some geodesics in the pseudoriemannian case may exhibit the same helical behavior as geodesics do in the Riemannian case. Different signatures on the Heisenberg group illustrate both possibilities (for example, $+-$ and $--$, respectively).

Remark 4.17 If $a_0 = 0$ or $x_0 = 0$, then the geodesic becomes $\gamma(t) = \exp(tx_0)$ or $\exp(ta_0)$, respectively. More generally, it follows from the geodesic equation (4.1) that if $\gamma(t)$ is the unique geodesic with $\dot{\gamma}(0) = a_0 + x_0$, then $\gamma(t) = \exp(t(a_0 + x_0))$ if and only if $\text{ad}_{x_0}^\dagger(a_0 + x_0) = 0$ if and only if $(a_0 + x_0) \perp [x_0, \mathfrak{n}]$. We note that this is also equivalent to $\text{ad}_{x_0}^\dagger(z_0 + v_0) = 0$ if and only if $(z_0 + v_0) \perp [x_0, \mathfrak{n}]$.

Proposition 4.18 *Let N be a simply connected, 2-step nilpotent Lie group with left-invariant metric tensor $\langle \cdot, \cdot \rangle$, and let γ be a geodesic with $\dot{\gamma}(0) = a \in \mathfrak{z}$. If $\text{ad}_\bullet^\dagger a = 0$, then there are no conjugate points along γ .*

Proof: It follows from Theorems 3.11 and 3.18 and Proposition 3.20 that all planes containing any $\dot{\gamma}(t)$ are homaloidal. Thus the Jacobi equation trivializes and the result follows. \square

In the Riemannian case [11, (3.10)] and in the timelike Lorentzian case (using [20, p.278, 23.Lemma] and an argument *mutatis mutandis*), the converse holds. It is not clear if it can be generalized much more than this.

5 Isometry group

We begin with a general result about nilpotent Lie groups.

Lemma 5.1 *Let N be a connected, nilpotent Lie group with a left-invariant metric tensor. Any isometry of N that is also an inner automorphism must be the identity map.*

Proof: Consider an isometry of N which is also the inner automorphism determined by $g \in N$. Then Ad_g is a linear isometry of \mathfrak{n} . The Lie group exponential map is surjective, so there exists $x \in \mathfrak{n}$ with $g = \exp(x)$. Then

$$\text{Ad}_g = e^{\text{ad}_x}.$$

Now ad_x is nilpotent so all its eigenvalues are 0. Thus all the eigenvalues of Ad_g are 1, so it is unipotent. It now follows that Ad_g is the identity. Thus g lies in the center of N and the isometry of N is the identity. \square

Now we give some general information about the isometries of N . Letting $\text{Aut}(N)$ denote the automorphism group of N and $I(N)$ the isometry group of N , set $O(N) = \text{Aut}(N) \cap I(N)$. In the Riemannian case,

$I(N) = O(N) \ltimes N$, the semidirect product where N acts as left translations. We have chosen the notation $O(N)$ to suggest an analogy with the pseudoeuclidean case in which this subgroup is precisely the (general, including reflections) pseudorthogonal group. According to Wilson [25], this analogy is good for any nilmanifold (not necessarily 2-step).

To see what is true about the isometry group in general, first consider the (left-invariant) splitting of the tangent bundle $TN = \mathfrak{z}N \oplus \mathfrak{v}N$.

Definition 5.2 *Denote by $I^{spl}(N)$ the subgroup of the isometry group $I(N)$ which preserves the splitting $TN = \mathfrak{z}N \oplus \mathfrak{v}N$. Further, let $I^{aut}(N) = O(N) \ltimes N$, where N acts by left translations.*

Proposition 5.3 *If N is a simply-connected, 2-step nilpotent Lie group with left-invariant metric tensor, then $I^{spl}(N) \leq I^{aut}(N)$.*

Proof: Similarly to the observation by Eberlein [11], it is now easy to check that the relevant part of Kaplan's proof for Riemannian N of H -type [13] is readily adapted and extended to our setting, with the proviso that it is easier to just replace his expressions $j(a)x$ with $\text{ad}_x^\dagger a$ instead of trying to convert to our j . \square

We shall give an example to show that $I^{spl} < I^{aut}$ is possible when $\mathfrak{U} \neq \{0\}$.

When the center is degenerate, the relevant group analogous to a pseudorthogonal group may be larger.

Proposition 5.4 *Let $\tilde{O}(N)$ denote the subgroup of $I(N)$ which fixes $1 \in N$. Then $I(N) \cong \tilde{O}(N) \ltimes N$, where N acts by left translations.* \square

The proof is obvious from the definition of \tilde{O} . It is also obvious that $O \leq \tilde{O}$. We shall give an example to show that $O < \tilde{O}$, hence $I^{aut} < I$, is possible when the center is degenerate.

Thus we have three groups of isometries, not necessarily equal in general: $I^{spl} \leq I^{aut} \leq I$. When the center is nondegenerate ($\mathfrak{U} = \{0\}$), the Ricci transformation is block-diagonalizable and the rest of Kaplan's proof using it now also works.

Corollary 5.5 *If the center is nondegenerate, then $I(N) = I^{spl}(N)$ whence $\tilde{O}(N) \cong O(N)$.* \square

We now begin to prepare for the promised examples. Recall the result in [24, 3.57(b)] that for any simply connected Lie group G with Lie algebra \mathfrak{g} ,

$$\text{Aut}(G) \cong \text{Aut}(\mathfrak{g}) : \alpha \longmapsto \alpha_*|_{\mathfrak{g}}.$$

Now, if α is also an isometry, then so is α_* . Conversely, any isometric automorphism of \mathfrak{g} comes from an isometric automorphism of G by means of left-invariance (or homogeneity). Therefore

$$O(N) = \text{Aut}(N) \cap I(N) \cong \text{Aut}(\mathfrak{n}) \cap O_q^p$$

for one of our simply connected groups N with signature (p, q) . So up to isomorphism, we may regard

$$O(N) \leq O_q^p.$$

Example 5.6 Any flat, simply connected N is a spaceform. (By Corollary 3.14 no other constant curvature is possible.) Thus the isometry group is known (up to isomorphism) and $\tilde{O}(N) \cong O_q^p$ for any such N of signature (p, q) .

For the rest of the examples, we need some additional preparations. Let \mathfrak{h}_3 denote the 3-dimensional Heisenberg algebra and consider the unique [9], 4-dimensional, 2-step nilpotent Lie algebra $\mathfrak{n}_4 = \mathfrak{h}_3 \times \mathbb{R}$, with basis $\{v, e, z, u\}$, structure equation $[v, e] = z$, and center $\mathfrak{z} = \llbracket z, u \rrbracket$.

A general automorphism of \mathfrak{n}_4 as a vector space, $f: \mathfrak{n}_4 \rightarrow \mathfrak{n}_4$, will be given with respect to the basis $\{v, e, z, u\}$ by a matrix

$$\begin{bmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{bmatrix}.$$

In order for f to be a Lie algebra automorphism of \mathfrak{n}_4 , it must preserve the center, and $[f(v), f(e)] = f(z)$ while the other products of images by f are zero; all this together implies that the matrix of f must have the form

$$\begin{bmatrix} a_1 & b_1 & 0 & 0 \\ a_2 & b_2 & 0 & 0 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & 0 & d_4 \end{bmatrix}$$

with $c_3 = a_1 b_2 - a_2 b_1 \neq 0$, $d_4 \neq 0$ and arbitrary a_3, a_4, b_3, b_4, d_3 .

Now the matrix of f can be decomposed as a product of matrices $M_1 \cdot M_2 \cdot M_3 \cdot M_4$, where

$$M_1 = \begin{bmatrix} a_1 & b_1 & 0 & 0 \\ a_2 & b_2 & 0 & 0 \\ A & B & c_3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad M_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & d_4 \end{bmatrix},$$

$$M_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_3/c_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad M_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ C & D & 0 & 1 \end{bmatrix},$$

with

$$A = \frac{a_3 d_4 - a_4 d_3}{d_4}, \quad B = \frac{b_3 d_4 - b_4 d_3}{d_4}, \quad C = \frac{a_4}{d_4}, \quad D = \frac{b_4}{d_4},$$

which are well defined because $c_3, d_4 \neq 0$. In this decomposition of f we have:

$M_1 \in \text{Aut}(\mathfrak{h}_3)$, and they fill out $\text{Aut}(\mathfrak{h}_3)$;

$M_2 \in \text{Aut}(\mathbb{R})$;

$M_3 \in M$ mixes part of the center but fixes v, e , and z ;

M_4 fixes z and u but the image of $\llbracket v, e \rrbracket$ is contained in $\llbracket v, e, u \rrbracket$.

The matrices M_4 form a group isomorphic to an abelian \mathbb{R}^2 . Note that for such an M_4 , $v \mapsto v + Cu$ and $e \mapsto e + Du$.

Proposition 5.7 *The decomposition of $\text{Aut}(\mathfrak{n}_4)$ is*

$$\text{Aut}(\mathfrak{n}_4) = \text{Aut}(\mathfrak{h}_3) \cdot \text{Aut}(\mathbb{R}) \cdot M \cdot \mathbb{R}^2.$$

□

Now we compute $O(N)$ for several simply-connected groups of interest. Specifically, we shall compute $O(N) \cong O(\mathfrak{n})$ for the flat and nondegenerate 3-dimensional Heisenberg groups H_3 and the 4-dimensional groups $N_4 \cong H_3 \times \mathbb{R}$ with all possible signatures.

Example 5.8 We begin with the form for automorphisms of \mathfrak{h}_3 on the basis $\{v, e, u\}$,

$$f = \begin{bmatrix} a_1 & b_1 & 0 \\ a_2 & b_2 & 0 \\ a_3 & b_3 & c_3 \end{bmatrix}$$

with $c_3 = a_1 b_2 - a_2 b_1 \neq 0$ and arbitrary a_3, b_3 . For the flat metric tensor we use

$$\eta = \begin{bmatrix} 0 & 0 & 1 \\ 0 & \bar{\varepsilon} & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

and we need to solve

$$f^T \eta f = \eta.$$

Using *Mathematica* to help, it is easy to see that we must have $b_1 = 0$, whence $a_1 \neq 0$ and thus the **Solve** command in fact does give all the solutions. We also obtain $a_1 = \pm 1$, so

$$f = \begin{bmatrix} \pm 1 & 0 & 0 \\ a_2 & 1 & 0 \\ \mp \bar{\varepsilon} a_2^2 / 2 & \mp \bar{\varepsilon} a_2 & \pm 1 \end{bmatrix} \quad (5.1)$$

with arbitrary a_2 . We note that the determinant is 1, so that this 1-parameter group lies in SO_2^1 or SO_1^2 according as $\bar{\varepsilon} = \pm 1$, respectively. With some additional work, this is seen to be a group of horocyclic translations (*HT* in [7]) subjected to a change of basis (rotation and reordering). (Comparing parameters, we have here $a_2 = -t\sqrt{2}$ there.) Note $\dim O(H_3) = 1$ while from Example 5.6 we have $\dim \tilde{O}(H_3) = 3$. We can identify the rest of \tilde{O} from an Iwasawa decomposition: O is the nilpotent part, so the rest of \tilde{O} consists of rotations and boosts.

Many groups with degenerate center have such a subgroup isometrically embedded.

Proposition 5.9 *Assume $\mathfrak{U} \neq \{0\} \neq \mathfrak{E}$ and let $u \in \mathfrak{U}$, $v \in \mathfrak{V}$, such that $\langle u, v \rangle = 1$ and $0 \neq j(u)v = e \in \mathfrak{E}$. Then $j(u)^2 = -I$ on $\llbracket v, e \rrbracket$ if and only if $\mathfrak{h} = \llbracket u, v, e \rrbracket$ is isometric to the 3-dimensional Heisenberg algebra with null center. Consequently, $H = \exp \mathfrak{h}$ is an isometrically embedded, flat subgroup of N .*

Proof: It is easy to see that $j(u)e = -v$; cf. Remarks 3.17. Then, on the one hand, $\langle [v, j(u)v], v \rangle = \langle [v, e], v \rangle$. On the other hand, $\langle [v, j(u)v], v \rangle = -\langle [j(u)v, v], v \rangle = -\langle \text{ad}_{j(u)v}^\dagger \iota u, v \rangle = -\langle \iota j(u)^2 v, v \rangle = \langle \iota v, v \rangle = \langle u, v \rangle = 1$. It follows that $[v, e] = u$. The converse follows from Example 2.5. \square

Corollary 5.10 *For any N containing such a subgroup,*

$$I^{spl}(N) < I^{aut}(N) < I(N).$$

\square

Unfortunately, this class does not include our flat groups of Theorem 3.15 in which $[\mathfrak{n}, \mathfrak{n}] \subseteq \mathfrak{U}$ and $\mathfrak{E} = \{0\}$. However, it does include many groups that do not satisfy $[\mathfrak{n}, \mathfrak{n}] \subseteq \mathfrak{U}$, such as the simplest quaternionic Heisenberg group of Example 2.6 *et seq.*

Remark 5.11 A direct computation shows that on this flat H_3 with null center, the only Killing fields with geodesic integral curves are the nonzero scalar multiples of u .

Similarly, we can do the 3-dimensional Heisenberg group with nondegenerate center (including certain definite cases). With respect to the orthonormal basis $\{e_1, e_2, z\}$ with structure equation $[e_1, e_2] = z$, we obtain

$$\tilde{O}(H_3) = O(H_3) \cong \begin{cases} O_2 & \text{if } \bar{\varepsilon}_1 \bar{\varepsilon}_2 = 1, \\ O_1^1 & \text{if } \bar{\varepsilon}_1 \bar{\varepsilon}_2 = -1 \end{cases}$$

of dimension two.

In general, we always can arrange to have e_1 and e_2 orthonormal, but we may not be able to keep the structure equation *and* have z a unit vector simultaneously. In the indefinite case, we always can have the z -axis orthogonal to the $e_1 e_2$ -plane. But in the definite case, the $e_1 e_2$ -plane need not be orthogonal to the z -axis. It is not yet clear how best to compute $O(H_3)$ in these cases.

Example 5.12 Now we consider the flat $N_4 \cong H_3 \times \mathbb{R}$ with null center. Here we have the basis $\{v_1, v_2, u_1, u_2\}$ of \mathfrak{n}_4 with structure equation $[v_1, v_2] = u_1$, the form for automorphisms

$$f = \begin{bmatrix} a_1 & b_1 & 0 & 0 \\ a_2 & b_2 & 0 & 0 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & 0 & d_4 \end{bmatrix}$$

with $c_3 = a_1 b_2 - a_2 b_1 \neq 0$, $d_4 \neq 0$, and arbitrary a_3, a_4, b_3, b_4, d_3 , and the flat metric tensor

$$\eta = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}.$$

Again we find $b_1 = 0$ and $a_1 \neq 0$ and **Solve** gives all solutions of $f^T \eta f = \eta$. We obtain

$$f = \begin{bmatrix} a_1 & 0 & 0 & 0 \\ a_2 & 1/a_1^2 & 0 & 0 \\ a_1^2 a_2 b_3 & b_3 & 1/a_1 & -a_1 a_2 \\ -a_1^3 b_3 & 0 & 0 & a_1^2 \end{bmatrix} \quad (5.2)$$

with $a_1 \neq 0$ and arbitrary a_2, b_3 . Again we note the determinant is 1 so this 3-parameter group lies in SO_2^2 . Note $\dim O(N_4) = 3$ while from Example 5.6 we have $\dim \tilde{O}(N_4) = 6$.

We consider three 1-parameter subgroups.

$$\begin{array}{ll} a_2 = b_3 = 0 & \text{I} \\ a_1 = 1, b_3 = 0 & \text{II} \\ a_1 = 1, a_2 = 0 & \text{III} \end{array}$$

We make a $2 + 2$ decomposition of our space into $\llbracket v_1, u_1 \rrbracket \oplus \llbracket v_2, u_2 \rrbracket$. If we think of $a_1 = \pm e^t$, then subgroup I is a boost by t , respectively $2t$, on the two subspaces. Subgroups II and III are both horocyclic translations on the two subspaces, and they commute with each other.

Again, we can see what the part of \tilde{O} outside O looks like from an Iwasawa decomposition $\tilde{O} = KAN \cong O_2^2$. Clearly, I is a subgroup of A while II and III are subgroups of N . Now O_2^2 has rank 2 so $\dim A = 2$. Since $\dim K = 2$, then $\dim N = 2$ also. Thus all the rotations and half of the boosts are the part of \tilde{O} outside O .

Many of our flat groups from Theorem 3.15 have such a subgroup isometrically embedded, as in fact do many others which are not flat.

Proposition 5.13 *Assume $[\mathfrak{V}, \mathfrak{V}] \subseteq \mathfrak{U}$ and $\dim \mathfrak{U} = \dim \mathfrak{V} \geq 2$. Let $u_1, u_2 \in \mathfrak{U}$ and $v_1, v_2 \in \mathfrak{V}$ be such that $\langle u_i, v_i \rangle = 1$. If, on $\llbracket v_1, v_2 \rrbracket$, $j(u_1)^2 = -I$ and $j(u) = 0$ for $u \notin \llbracket u_1 \rrbracket$, then $\llbracket u_1, u_2, v_1, v_2 \rrbracket \cong \mathfrak{h}_3 \times \mathbb{R}$.*

Proof: We may assume without loss of generality that $j(u_1)v_1 = v_2$. Then a straightforward calculation shows that $\llbracket v_1, v_2 \rrbracket = u_1$ and the result follows. \square

Corollary 5.14 *For any N containing such a subgroup,*

$$I^{spl}(N) < I^{aut}(N) < I(N). \quad \square$$

Example 5.15 Next we consider some N_4 with only partially degenerate center. Here we have the basis $\{v, e, z, u\}$ of \mathfrak{n}_4 with structure equation $[v, e] = z$ and signature $(+\bar{\varepsilon}\varepsilon-)$. We take f again as above, and the metric tensor

$$\eta = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & \bar{\varepsilon} & 0 & 0 \\ 0 & 0 & \varepsilon & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

Applying `Solve` to $f^T \eta f = \eta$ directly is quite wasteful this time, producing many solutions with $\varepsilon = 0$ which must be discarded, and producing duplicates of the 4 correct solutions. After removing the rubbish, however, it is easy to check that there are no more again. Thus we obtain

$$f = \begin{bmatrix} \pm 1 & 0 & 0 & 0 \\ a_2 & \pm 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \mp \bar{\varepsilon} a_2^2 / 2 & \mp \bar{\varepsilon} a_2 & 0 & \pm 1 \end{bmatrix}$$

or (5.3)

$$f = \begin{bmatrix} \pm 1 & 0 & 0 & 0 \\ a_2 & \mp 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ \mp \bar{\varepsilon} a_2^2 / 2 & \mp \bar{\varepsilon} a_2 & 0 & \pm 1 \end{bmatrix}$$

with arbitrary a_2 . Now we have the determinant ± 1 for these 1-parameter subgroups, and both are contained in O_3^{1+} , O_2^{2+} , or O_1^{3+} , according to the choices for $\bar{\varepsilon}$ and ε , respectively $\bar{\varepsilon} = \varepsilon = 1$, $\bar{\varepsilon} \neq \varepsilon$, or $\bar{\varepsilon} = \varepsilon = -1$. Again, these are all horocyclic translations as found in [7], modulo change of basis.

This yields $\dim O(N_4) = 1$, while in the next example we shall see that $\dim \widetilde{O}(N_4) = 2$.

Since N is complete, so are all Killing fields. It follows [20, p. 255] that the set of all Killing fields on N is the Lie algebra of the (full) isometry group $I(N)$. Also, the 1-parameter groups of isometries constituting the flow of any Killing field are global. Thus integration of a Killing field produces (global, not just local) isometries of N .

Example 5.16 Consider again the unique [9], nonabelian, 2-step nilpotent Lie algebra \mathfrak{n}_4 of dimension 4. We take a metric tensor so that the center

is only partially degenerate, as in the preceding Example. Thus we choose the basis $\{v, e, z, u\}$ with structure equation $[v, e] = z$ and nontrivial inner products

$$\langle v, u \rangle = 1, \quad \langle e, e \rangle = \bar{\varepsilon}, \quad \langle z, z \rangle = \varepsilon.$$

This basis is partly null and partly orthonormal. The associated simply connected group N_4 is isomorphic to $H_3 \times \mathbb{R}$ and we realize it as the real matrices

$$\begin{bmatrix} 1 & x & z & 0 & 0 \\ 0 & 1 & y & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & w \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \quad (5.4)$$

so this group is diffeomorphic to \mathbb{R}^4 . Take

$$v = \frac{\partial}{\partial x}, \quad e = \frac{\partial}{\partial y} + x \frac{\partial}{\partial z}, \quad z = \frac{\partial}{\partial z}, \quad u = \frac{\partial}{\partial w}, \quad (5.5)$$

which realizes the structure equation in the given representation. (The double use of z should not cause any confusion in context.) The nontrivial adjoint maps are

$$\text{ad}_v^\dagger z = \varepsilon \bar{\varepsilon} e \quad \text{and} \quad \text{ad}_e^\dagger z = -\varepsilon u$$

so the nontrivial covariant derivatives are

$$\begin{aligned} \nabla_v e &= -\nabla_e v = \frac{1}{2} z, \\ \nabla_z v &= \nabla_v z = -\frac{1}{2} \varepsilon \bar{\varepsilon} e, \\ \nabla_z e &= \nabla_e z = \frac{1}{2} \varepsilon u. \end{aligned}$$

Remark 5.17 The space N_4 is not flat; in fact, a direct computation shows that

$$\langle R(z, v)v, z \rangle = \frac{1}{4} \bar{\varepsilon} \quad \text{and} \quad \langle R(v, e)e, v \rangle = -\frac{3}{4} \varepsilon.$$

We consider Killing fields X and write $X = f^1 u + f^2 z + f^3 v + f^4 e$. We denote partial derivatives by subscripts. From Killing's equation we obtain the system.

$$f_x^1 = f_z^2 = f_w^3 = 0 \quad (5.6)$$

$$f_w^1 + f_x^3 = 0 \quad (5.7)$$

$$\varepsilon f_w^2 + f_z^3 = 0 \quad (5.8)$$

$$f_y^4 + x f_z^4 = 0 \quad (5.9)$$

$$\varepsilon f_z^1 + f_x^2 + f^4 = 0 \quad (5.10)$$

$$\bar{\varepsilon} f_x^4 + f_y^1 + x f_z^1 = 0 \quad (5.11)$$

$$\bar{\varepsilon} f_w^4 + f_y^3 + x f_z^3 = 0 \quad (5.12)$$

$$\varepsilon \bar{\varepsilon} f_z^4 + f_y^2 = f^3 \quad (5.13)$$

Simple considerations of second derivatives yield these relations.

$$\begin{aligned} f_x^1 &= f_{ww}^1 = f_{wy}^1 = f_{wz}^1 = 0 \\ f_z^2 &= f_{ww}^2 = f_{wx}^2 = 0 \\ f_w^3 &= f_{xx}^3 = f_{xz}^3 = f_{zz}^3 = 0 \\ f_w^4 &= f_{xz}^4 = 0 \end{aligned}$$

From the first we get $f_w^1 = A$, and then from (5.7) $f_x^3 = -A$, so in particular $f_{xy}^3 = 0$. Then (5.12) and $f_w^4 = f_{xy}^3 = f_{xz}^3 = 0$ imply $f_z^3 = 0$ and therefore $f_y^3 = 0$ also. Hence,

$$f^3 = -Ax + C_1.$$

Also, (5.8) and $f_z^3 = 0$ imply $f_w^2 = 0$, and (5.13) and $f_{zx}^4 = 0$ imply $f_{yx}^2 = f_x^3 = -A$. It then follows that

$$\begin{aligned} f^1 &= Aw + F^1(y, z), \\ f^2 &= F^2(x, y), \quad \text{with } F_{xy}^2 = -A, \\ f^3 &= -Ax + C_1, \\ f^4 &= f^4(x, y, z), \quad \text{with } f_{xz}^4 = 0. \end{aligned}$$

Now, using (5.11) and $f_{xz}^4 = 0$, after a few steps we obtain $F_z^1 = C_2$ whence

$$F^1(y, z) = C_2 z + G^1(y).$$

Then (5.10) gives us $f_z^4 = -\varepsilon f_{zz}^1 = 0$ so

$$f^4 = -\varepsilon C_2 - F_x^2(x, y).$$

From (5.9) and $f_z^4 = 0$ we get $0 = f_y^4 = -F_{xy}^2 = A$ whence

$$A = 0.$$

From (5.13) and $f_z^4 = 0$ we get $f_y^2 = F_y^2 = C_1$ whence $F^2(x, y) = C_1 y + G^2(x)$; hence

$$\begin{aligned} f^1 &= C_2 z + G^1(y), \\ f^2 &= C_1 y + G^2(x), \\ f^3 &= C_1, \\ f^4 &= -\varepsilon C_2 - G_x^2(x). \end{aligned}$$

Finally, (5.11) implies $-\bar{\varepsilon} G_{xx}^2 + G_y^1 + C_2 x = 0$ so $-\bar{\varepsilon} G_{xxx}^2 = -C_2$ and hence

$$\begin{aligned} G_{xxx}^2 &= \bar{\varepsilon} C_2, \\ G_{xx}^2 &= \bar{\varepsilon} C_2 x + C_3, \\ G_x^2 &= \frac{\bar{\varepsilon} C_2}{2} x^2 + C_3 x + C_4, \\ G^2 &= \frac{\bar{\varepsilon} C_2}{6} x^3 + \frac{C_3}{2} x^2 + C_4 x + C_5. \end{aligned}$$

Then $G_y^1 = \bar{\varepsilon} G_{xx}^2 - C_2 x = \bar{\varepsilon} C_3$ whence

$$G^1(y) = \bar{\varepsilon} C_3 y + C_6.$$

and we are done.

$$\begin{aligned} f^1 &= \bar{\varepsilon} C_3 y + C_2 z + C_6 \\ f^2 &= \frac{\bar{\varepsilon} C_2}{6} x^3 + \frac{C_3}{2} x^2 + C_4 x + C_1 y + C_5 \\ f^3 &= C_1 \\ f^4 &= -\frac{\bar{\varepsilon} C_2}{2} x^2 - C_3 x - (\varepsilon C_2 + C_4) \end{aligned}$$

Therefore, $\dim I(N_4) = 6$.

The Killing vector fields with $X(0) = 0$ arise for $C_1 = C_5 = C_6 = 0$ and $C_4 = -\varepsilon C_2$, so their coefficients are

$$\begin{aligned} f^1 &= \bar{\varepsilon} C_3 y + C_2 z, \\ f^2 &= \frac{\bar{\varepsilon} C_2}{6} x^3 + \frac{C_3}{2} x^2 - \varepsilon C_2 x, \\ f^3 &= 0, \\ f^4 &= -\frac{\bar{\varepsilon} C_2}{2} x^2 - C_3 x. \end{aligned}$$

Therefore $\dim \tilde{O}(N_4) = 2$.

Proposition 5.18 *The Killing vector fields X with $X(0) = 0$ are*

$$X = -\left(\frac{\bar{\varepsilon}C_2}{2}x^2 + C_3x\right)\frac{\partial}{\partial y} - \left(\frac{\bar{\varepsilon}C_2}{3}x^3 + \frac{C_3}{2}x^2 + \varepsilon C_2x\right)\frac{\partial}{\partial z} \\ + (\bar{\varepsilon}C_3y + C_2z)\frac{\partial}{\partial w}.$$

Their integral curves are the solutions of

$$\begin{aligned}\dot{x}(t) &= 0, \\ \dot{y}(t) &= -\frac{\bar{\varepsilon}C_2}{2}x^2 - C_3x, \\ \dot{z}(t) &= -\frac{\bar{\varepsilon}C_2}{3}x^3 - \frac{C_3}{2}x^2 - \varepsilon C_2x, \\ \dot{w}(t) &= \bar{\varepsilon}C_3y + C_2z,\end{aligned}$$

and the solutions with initial condition (x_0, y_0, z_0, w_0) are

$$\begin{aligned}x(t) &= x_0, \\ y(t) &= -\left(\frac{\bar{\varepsilon}C_2}{2}x_0^2 + C_3x_0\right)t + y_0, \\ z(t) &= -\left(\frac{\bar{\varepsilon}C_2}{3}x_0^3 + \frac{C_3}{2}x_0^2 + \varepsilon C_2x_0\right)t + z_0, \\ w(t) &= -\left(\frac{\bar{\varepsilon}C_2^2}{3}x_0^3 + C_2C_3x_0^2 + (\varepsilon C_2^2 + \bar{\varepsilon}C_3^2)x_0\right)\frac{t^2}{2} + (\bar{\varepsilon}C_3y_0 + C_2z_0)t + w_0.\end{aligned}$$

Thus the 1-parameter group Φ_t of isometries is given by

$$\begin{aligned}\Phi_t^1 &= x, \\ \Phi_t^2 &= -\left(\frac{\bar{\varepsilon}C_2}{2}x^2 + C_3x\right)t + y, \\ \Phi_t^3 &= -\left(\frac{\bar{\varepsilon}C_2}{3}x^3 + \frac{C_3}{2}x^2 + \varepsilon C_2x\right)t + z, \\ \Phi_t^4 &= -\left(\frac{\bar{\varepsilon}C_2^2}{3}x^3 + C_2C_3x^2 + (\varepsilon C_2^2 + \bar{\varepsilon}C_3^2)x\right)\frac{t^2}{2} + (\bar{\varepsilon}C_3y + C_2z)t + w,\end{aligned}$$

and its Jacobian is

$$\Phi_{t*} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -(\bar{\varepsilon}C_2x + C_3)t & 1 & 0 & 0 \\ -(\bar{\varepsilon}C_2x^2 + C_3x + \varepsilon C_2)t & 0 & 1 & 0 \\ -\frac{1}{2}(\bar{\varepsilon}C_2^2x^2 + 2C_2C_3x + (\varepsilon C_2^2 + \bar{\varepsilon}C_3^2))t^2 & \bar{\varepsilon}C_3t & C_2t & 1 \end{bmatrix}.$$

Hence

$$\begin{aligned}\Phi_{t*}v &= v - (C_3 + \bar{\varepsilon}C_2x)te - \varepsilon C_2tz - (\bar{\varepsilon}C_2^2x^2 + 2C_2C_3x + \varepsilon C_2^2 + \bar{\varepsilon}C_3^2)\frac{t^2}{2}u, \\ \Phi_{t*}e &= e + (\bar{\varepsilon}C_3 + C_2x)tu, \\ \Phi_{t*}z &= z + C_2tu, \\ \Phi_{t*}u &= u.\end{aligned}$$

Evaluating at the identity element of N_4 , $(0, 0, 0, 0) \in \mathbb{R}^4$,

$$\begin{aligned}\Phi_{t*}v &= v - C_3te - \varepsilon C_2tz - \frac{\varepsilon C_2^2 + \bar{\varepsilon}C_3^2}{2}t^2u, \\ \Phi_{t*}e &= e + \bar{\varepsilon}C_3tu, \\ \Phi_{t*}z &= z + C_2tu, \\ \Phi_{t*}u &= u.\end{aligned}$$

Note that $C_2 = 0$ corresponds to (5.3). On the basis $\{v, e, z, u\}$, the matrix of such a Φ_{t*} (at the identity) is

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ -C_3t & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\frac{1}{2}\bar{\varepsilon}C_3^2t^2 & \bar{\varepsilon}C_3t & 0 & 1 \end{bmatrix}.$$

Proposition 5.19 $\Phi_{t*} \in I^{spl}(N_4)$ if and only if $C_2 = C_3 = 0$. $\Phi_{t*} \in I^{aut}(N_4)$ if and only if $C_2 = 0$. Thus $I^{spl}(N_4) < I^{aut}(N_4) < I(N_4)$. \square

Finally, we give the matrix for a Φ_{t*} (at the identity) on the basis $\{v, e, z, u\}$, as in (5.3) but now with $C_3 = 0$ instead. We find

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -\varepsilon C_2t & 0 & 1 & 0 \\ -\frac{1}{2}\varepsilon C_2^2t^2 & 0 & C_2t & 1 \end{bmatrix}.$$

And yet again, these are horocyclic translations as found in [7], modulo change of basis.

6 Lattices and Periodic Geodesics

In this section, we assume that N is rational and let Γ be a lattice in N . Since N is 2-step nilpotent, it has nice generating sets of Γ ; cf. [10, (5.3)].

Proposition 6.1 *If N is a simply connected, 2-step nilpotent Lie group of dimension n with lattice Γ and with center Z of dimension m , then there exists a canonical generating set $\{\varphi_1, \dots, \varphi_n\}$ such that $\{\varphi_1, \dots, \varphi_m\}$ generate $\Gamma \cap Z$. In particular, $\Gamma \cap Z$ is a lattice in Z .*

From this, formula (1.1), and [22, Prop. 2.17], one obtains as in [11, (5.3)]

Corollary 6.2 *Let N be a simply connected, 2-step nilpotent Lie group with lattice Γ and let $\pi : \mathfrak{n} \rightarrow \mathfrak{v}$ denote the projection. Then*

1. $\log \Gamma \cap \mathfrak{z}$ is a vector lattice in \mathfrak{z} ;
2. $\pi(\log \Gamma)$ is a vector lattice in \mathfrak{v} ;
3. $\Gamma \cap Z = Z(\Gamma)$. □

Here, we used the splitting $\mathfrak{n} = \mathfrak{u} \oplus \mathfrak{z} \oplus \mathfrak{v} \oplus \mathfrak{e}$ from Section 2 with $\mathfrak{z} = \mathfrak{u} \oplus \mathfrak{z}$ and $\mathfrak{v} = \mathfrak{v} \oplus \mathfrak{e}$.

Thus to the compact nilmanifold $\Gamma \backslash N$ we may associate two flat (possibly degenerate) tori; cf. [11, p. 644].

Definition 6.3 *With notation as preceding,*

$$\begin{aligned} T_{\mathfrak{z}} &= \mathfrak{z} / (\log \Gamma \cap \mathfrak{z}), \\ T_{\mathfrak{v}} &= \mathfrak{v} / \pi(\log \Gamma). \end{aligned}$$

Observe that $\dim T_{\mathfrak{z}} + \dim T_{\mathfrak{v}} = \dim \mathfrak{z} + \dim \mathfrak{v} = \dim \mathfrak{n}$.

Next, we apply Theorem 3 from [21] to our situation. Let T^m denote the m -torus as usual.

Theorem 6.4 *Let $m = \dim \mathfrak{z}$ and $n = \dim \mathfrak{v}$. Then $\Gamma \backslash N$ is a principal T^m -bundle over T^n .*

The model fiber T^m can be given a geometric structure from its closed embedding in $\Gamma \backslash N$; we denote this geometric m -torus by T_F . Similarly, we wish to provide the base n -torus with a geometric structure so that the projection $p_B : \Gamma \backslash N \rightarrow T_B$ is the appropriate generalization of a pseudoriemannian submersion [20] to (possibly) degenerate spaces. Observe that the splitting $\mathfrak{n} = \mathfrak{z} \oplus \mathfrak{v}$ induces splittings $TN = \mathfrak{z}N \oplus \mathfrak{v}N$ and $T(\Gamma \backslash N) = \mathfrak{z}(\Gamma \backslash N) \oplus \mathfrak{v}(\Gamma \backslash N)$, and that p_{B*} just mods out $\mathfrak{z}(\Gamma \backslash N)$. Examining the definition on page 212 of [20], we see that the key is to construct the geometry of T_B by defining

$$p_{B*} : \mathfrak{v}_{\eta}(\Gamma \backslash N) \rightarrow T_{p_B(\eta)}(T_B) \text{ for each } \eta \in \Gamma \backslash N \text{ is an isometry} \quad (6.1)$$

and

$$\nabla_{p_{B*}x}^{T_B} p_{B*}y = p_{B*}(\pi \nabla_x y) \text{ for all } x, y \in \mathfrak{v} = \mathfrak{V} \oplus \mathfrak{E}, \quad (6.2)$$

where $\pi : \mathfrak{n} \rightarrow \mathfrak{v}$ is the projection. Then the rest of the results of pages 212–213 in [20] will continue to hold, provided that sectional curvature is replaced by the numerator of the sectional curvature formula in his 47. Theorem at least when elements of \mathfrak{V} are involved:

$$\langle R_{T_B}(p_{B*}x, p_{B*}y)p_{B*}y, p_{B*}x \rangle = \langle R_{\Gamma \setminus N}(x, y)y, x \rangle + \frac{3}{4}\langle [x, y], [x, y] \rangle. \quad (6.3)$$

Now p_B will be a pseudoriemannian submersion in the usual sense if and only if $\mathfrak{U} = \mathfrak{V} = \{0\}$, as is always the case for Riemannian spaces.

In the Riemannian case, Eberlein showed that $T_F \cong T_3$ and $T_B \cong T_v$. Observe that it follows from Theorem 3.19, Proposition 3.20, and (6.3) that T_B is flat in general only if N has a nondegenerate center or is flat.

Proposition 6.5 *If $v, v' \in \mathfrak{V}$ and $e, e' \in \mathfrak{E}$, then*

$$\begin{aligned} \langle R_{T_B}(v, e)e, v \rangle &= \frac{1}{4}\langle j(\iota v)e, j(\iota v)e \rangle, \\ \langle R_{T_B}(v, v')v', v \rangle &= \frac{1}{2}\langle j(\iota v)v', j(\iota v')v \rangle - \langle j(\iota v)v, j(\iota v')v' \rangle \\ &\quad + \frac{1}{4}\left(\langle j(\iota v')v, j(\iota v')v \rangle + \langle j(\iota v)v', j(\iota v)v' \rangle\right), \\ K_{T_B}(e, e') &= 0. \end{aligned}$$

Here we have suppressed p_{B*} on the left-hand sides for simplicity. □

Our quaternionic Heisenberg group (Examples 2.6 and 3.22) provides a simple example of a T_B that is not flat. Indeed, all sectional curvature numerators vanish except $\langle R_{T_B}(v_1, v_2)v_2, v_1 \rangle = -\bar{\varepsilon}_1$, where we have again suppressed p_{B*} for simplicity.

Remark 6.6 Observe that the torus T_B may be decomposed into a topological product $T_E \times T_V$ in the obvious way. It is easy to check that T_E is flat and isometric to $(\log \Gamma \cap \mathfrak{E}) \setminus \mathfrak{E}$, and that T_V has a linear connection not coming from a metric and not flat in general. Moreover, the geometry of the product is “twisted” in a certain way. It would be interesting to determine which tori could appear as such a T_V and how.

We wish to show that the geometry of the fibers T_F is that of T_3 . Thus we now consider the submersion $\Gamma \setminus N \twoheadrightarrow T_B$ and realize the model fiber T^m as $(\Gamma \cap Z) \setminus Z$ considered as tori only, without geometry.

Definition 6.7 Let $p_N : N \rightarrow \Gamma \backslash N$ and $p_Z : Z \rightarrow T^m = (\Gamma \cap Z) \backslash Z$ be the natural projections. Define $F : T^m \rightarrow I(\Gamma \backslash N)$ by $F(p_Z(z))(p_N(n)) = p_N(zn)$ for all $z \in Z$ and $n \in N$.

Proposition 6.8 F is a smooth isomorphism of groups $T^m \cong I_0^{spl}(\Gamma \backslash N)$, where the subscript 0 denotes the identity component.

Proof: We follow Eberlein [11, (5.4)]. It is easy to check that F is a well-defined, smooth, injective homomorphism with image in $I_0^{spl}(\Gamma \backslash N)$. Thus we need only show that F is surjective.

Let $\psi \in I_0^{spl}(\Gamma \backslash N)$ and let ϕ_t be a path from $1 = \phi_0$ to $\psi = \phi_1$. The covering map p_N has the homotopy lifting property, so choose a lifting $\tilde{\phi}_t$ as a path in $I_0^{spl}(N)$. Then for all $g \in \Gamma$, it follows that $p_N(\tilde{\phi}_t L_g \tilde{\phi}_t^{-1}) = p_N$ for all t . Hence for each $t \in [0, 1]$, there exists $g_t \in \Gamma$ such that $\tilde{\phi}_t L_{g_t} \tilde{\phi}_t^{-1} = L_{g_t}$. Since $g_0 = g$ and Γ is a discrete group, it follows that $g_t = g$ for every t , so L_g commutes with $\tilde{\phi}_t$ for every $g \in \Gamma$ and $t \in [0, 1]$.

From Proposition 5.3, there exist $n_t \in N$ and $a_t \in O(N)$ such that $\tilde{\phi}_t = L_{n_t} a_t$ for all t . Now, every L_g commutes with every $\tilde{\phi}_t$, so $a_t(g) = n_t^{-1} g n_t$ for all t and g . Extension from lattices is unique [22, Thm. 2.11], so $a_t = \text{Ad}_{n_t^{-1}}$. By Lemma 5.1, a_t is the identity and $n_t \in Z$ for all t . Thus $\tilde{\phi}_1 = L_{n_1}$, so from the definition of $\tilde{\phi}_t$ we obtain $p_N L_{n_1} = p_N \tilde{\phi}_1 = \phi_1 p_N = \psi p_N$. But this means $F(p_Z(n_1)) = \psi$. \square

Corollary 6.9 $I_0^{spl}(\Gamma \backslash N)$ acts freely on $\Gamma \backslash N$ with complete, flat, totally geodesic orbits.

Proof: By Theorem 6.4, we may identify $I_0^{spl}(\Gamma \backslash N)$ as the group of the principal bundle $\Gamma \backslash N \twoheadrightarrow T_B$, so it acts freely on the total space.

Since p_N is a local isometry and the Z -orbits in N are complete, flat, and totally geodesic from Example 4.3, it follows (using the identification $T^m = (\Gamma \cap Z) \backslash Z$ *supra*) that the $I_0^{spl}(\Gamma \backslash N)$ -orbits are complete, flat, and totally geodesic. \square

Theorem 6.10 Let N be a simply connected, 2-step nilpotent Lie group with lattice Γ , a left-invariant metric tensor, and tori as in the discussion following Theorem 6.4. The fibers T_F of the (generalized) pseudoriemannian submersion $\Gamma \backslash N \twoheadrightarrow T_B$ are isometric to T_3 .

Proof: We follow the proof of Eberlein [11, (5.5), item 2]. For each $n \in N$, define $\psi_n = p_N L_n \exp : \mathfrak{z} \rightarrow \Gamma \backslash N$, and note that it is a local isometry.

Clearly, $\psi_n(z) = \psi_n(z')$ if and only if $z' = z + \log g$ for some $g \in \Gamma \cap Z$. Hence ψ_n induces an isometric embedding $\tilde{\psi}_n : T_3 \rightarrow \Gamma \backslash N$. That the image is the $I_0^{spl}(\Gamma \backslash N)$ -orbit of $p_N(n)$ follows from the proof of Corollary 6.9. \square

Corollary 6.11 *If in addition the center Z of N is nondegenerate, then T_B is isometric to T_v .* \square

The proof is essentially the same as the appropriate parts of the proof of (5.5) in [11] and we omit it.

We recall that elements of N can be identified with elements of the isometry group $I(N)$: namely, $n \in N$ is identified with the isometry $\phi = L_n$ of left translation by n . We shall abbreviate this by writing $\phi \in N$.

Definition 6.12 *We say that $\phi \in N$ translates the geodesic γ by ω if and only if $\phi\gamma(t) = \gamma(t + \omega)$ for all t . If γ is a unit-speed geodesic, we say that ω is a period of ϕ .*

Recall that unit speed means that $|\dot{\gamma}| = |\langle \dot{\gamma}, \dot{\gamma} \rangle|^{\frac{1}{2}} = 1$. Since there is no natural normalization for null geodesics, we do not define periods for them. In the Riemannian case and in the timelike Lorentzian case in strongly causal spacetimes [2], unit-speed geodesics are parameterized by arclength and this period is a translation distance. If ϕ belongs to a lattice Γ , it is the length of a closed geodesic in $\Gamma \backslash N$.

Remark 6.13 Note that it follows from Corollary 1.2 that if $\phi = \exp(a^* + x^*)$ translates a geodesic γ with $\gamma(0) = 1 \in N$, then $a^* + x^*$ and $\dot{\gamma}(0)$ are of the same causal character.

In general, recall that if γ is a geodesic in N and if $p_N : N \twoheadrightarrow \Gamma \backslash N$ denotes the natural projection, then $p_N\gamma$ is a periodic geodesic in $\Gamma \backslash N$ if and only if some $\phi \in \Gamma$ translates γ . We say *periodic* rather than *closed* here because in pseudoriemannian spaces it is possible for a null geodesic to be closed but not periodic. If the space is geodesically complete or Riemannian, however, then this does not occur (*cf.* [20], p. 193); the former is in fact the case for our 2-step nilpotent Lie groups by Theorem 1.5. Further recall that free homotopy classes of closed curves in $\Gamma \backslash N$ correspond bijectively with conjugacy classes in Γ .

Definition 6.14 *Let \mathcal{C} denote either a nontrivial, free homotopy class of closed curves in $\Gamma \backslash N$ or the corresponding conjugacy class in Γ . We define $\wp(\mathcal{C})$ to be the set of all periods of periodic unit-speed geodesics that belong to \mathcal{C} .*

In the Riemannian case, this is the set of lengths of closed geodesics in \mathcal{C} , frequently denoted by $\ell(\mathcal{C})$.

Definition 6.15 *The period spectrum of $\Gamma \backslash N$ is the set*

$$\text{spec}_{\wp}(\Gamma \backslash N) = \bigcup_{\mathcal{C}} \wp(\mathcal{C}),$$

where the union is taken over all nontrivial, free homotopy classes of closed curves in $\Gamma \backslash N$.

In the Riemannian case, this is the length spectrum $\text{spec}_{\ell}(\Gamma \backslash N)$.

Example 6.16 Similar to the Riemannian case, we can compute the period spectrum of a flat torus $\Gamma \backslash \mathbb{R}^m$, where Γ is a lattice (of maximal rank, isomorphic to \mathbb{Z}^m). Using calculations related to those of [5, pp.146–8] in an analogous way as for finding the length spectrum of a Riemannian flat torus, we easily obtain

$$\text{spec}_{\wp}(\Gamma \backslash \mathbb{R}^m) = \{|g| \neq 0 \mid g \in \Gamma\}.$$

It is also easy to see that the nonzero d'Alembertian spectrum is related to the analogous set produced from the dual lattice Γ^* as multiples by $\pm 4\pi^2$, almost as in the Riemannian case.

As in this example, simple determinacy of periods of unit-speed geodesics helps make calculation of the period spectrum possible purely in terms of $\log \Gamma \subseteq \mathfrak{n}$. (See Theorem 6.25 for another example.) Thus we begin with the following observation.

Proposition 6.17 *Let $\phi = \exp(a^* + v^* + e^*)$ translate the unit-speed geodesic γ by $\omega > 0$. If $v^* \neq 0$, then the period ω is simply determined.*

Proof: We may assume $\gamma(0) = 1 \in N$ and $\dot{\gamma}(0) = a_0 + v_0 + e_0$. From Theorem 4.11, $v^* = v(\omega) = \omega v_0$. □

In attempting to calculate the period spectrum then, we can focus our attention on those cases where the \mathfrak{V} -component is zero.

From now on, we assume that N is a simply connected, 2-step nilpotent Lie group with left-invariant pseudoriemannian metric tensor $\langle \cdot, \cdot \rangle$. Note that non-null geodesics may be taken to be of unit speed. Most nonidentity elements of N translate some geodesic, but not necessarily one of unit speed; cf. [11, (4.2)].

Proposition 6.18 *Let N be a simply connected, 2-step nilpotent Lie group with left-invariant metric tensor $\langle \cdot, \cdot \rangle$ and $\phi \in N$ not the identity. Write $\log \phi = a^* + x^* \in \mathfrak{z} \oplus \mathfrak{v}$ and assume that $x^* \perp [x^*, \mathfrak{n}]$. Let a' be the component of a^* orthogonal to $[x^*, \mathfrak{n}]$ in \mathfrak{z} and choose $\xi \in \mathfrak{n}$ such that $a' = a^* + [x^*, \xi]$. Set $\omega^* = |a' + x^*|$ if $a' + x^*$ is not null and set $\omega^* = 1$ otherwise. Then ϕ translates the geodesic*

$$\gamma(t) = \exp(\xi) \exp\left(\frac{t}{\omega^*}(a' + x^*)\right)$$

by ω^ , and γ is of unit speed if $a' + x^*$ is not null.*

Proof: Let $n = \exp(\xi)$ and set $\phi^* = n^{-1}\phi n = \exp(a' + x^*)$ and $\gamma^*(t) = n^{-1}\gamma(t)$. Then $\phi\gamma(t) = \gamma(t + \omega^*)$ is equivalent to $\phi^*\gamma^*(t) = \gamma^*(t + \omega^*)$, and the latter is routine to verify using (1.1).

Now γ is a geodesic if and only if γ^* is, and it is easy to check directly that $\nabla_{a'+x^*}(a' + x^*) = 0$ is equivalent to $\langle a' + x^*, [x^*, \mathfrak{n}] \rangle = 0$. \square

Note that the \mathfrak{U} components of a^* and a' in fact coincide. Also note that if we further decompose $x^* = v^* + e^*$, the result applies to every nonidentity element ϕ with $v^* = 0$. In particular, when the center is nondegenerate this is every nonidentity element.

Corollary 6.19 *When \mathfrak{n} is nonsingular and $\phi \notin Z$, we may take $a' = 0$ in Proposition 6.18.*

Proof: Because then $a^* \in [x^*, \mathfrak{n}]$ and $x^* \neq 0$. \square

Now we give some general criteria for an element ϕ to translate a geodesic γ ; cf. [11, (4.3)]. We use a J as in the passage following Definition 4.7, and x_1, x_2, y_1 , and y_2 as given just before Theorem 4.11.

Proposition 6.20 *Let $\phi \in N$ and write $\phi = \exp(a^* + x^*)$ for suitable elements $a^* \in \mathfrak{z}$ and $x^* \in \mathfrak{v}$. Let γ be a geodesic with $\gamma(0) = n \in N$ and $\gamma(\omega) = \phi n$, and let $\dot{\gamma}(0) = L_{n*}(a_0 + x_0)$ for suitable elements $a_0 \in \mathfrak{z}$ and $x_0 = v_0 + e_0 \in \mathfrak{v}$. Let $n^{-1}\gamma(t) = \exp(a(t) + x(t))$ where $a(t) \in \mathfrak{z}$ and $x(t) \in \mathfrak{v}$ for all $t \in \mathbb{R}$ and $a(0) = x(0) = 0$. Then the following are equivalent:*

1. $\left. \begin{aligned} x(t + \omega) &= x(t) + x^* \\ a(t + \omega) &= a(t) + a^* + \frac{1}{2}[x^*, x(t)] \end{aligned} \right\} \text{ for all } t \in \mathbb{R} \text{ and some } \omega > 0;$
2. $\gamma(t + \omega) = \phi\gamma(t)$ for all $t \in \mathbb{R}$ and some $\omega > 0$;

3. $e^{\omega J}$ fixes $e_1 + y_1 + x_2 = e_0 + y_1 + J^{-1}y_2$.

Proof: As before, we may assume without loss of generality that $n = 1 \in N$. Items 1 and 2 are equivalent by formula (1.1). The following lemma shows that item 1 implies item 3.

Lemma 6.21 *As in the preamble to Theorem 4.11, write \mathfrak{E} as an orthogonal direct sum $\mathfrak{E}_1 \oplus \mathfrak{E}_2$, where $\mathfrak{E}_1 = \ker J$, and use x_1, x_2, y_1 , and y_2 as given there. Then $x^* = \omega x_1 + \frac{1}{2}\omega^2 y_1$ and $e^{\omega J}$ fixes x_2 .*

Proof: By Theorem 4.11, we have $x(k\omega) = k\omega x_1 + (e^{k\omega J} - I)J^{-1}x_2 + \frac{1}{2}k^2\omega^2 y_1$ for every positive integer k . By induction, from item 1 in the statement of the proposition we obtain $x(k\omega) = kx^*$ for every k . Decomposing $x^* = v^* + e_1^* + e_2^* \in \mathfrak{V} \oplus \mathfrak{E}_1 \oplus \mathfrak{E}_2$, this yields $v^* = \omega v_0$, $e_1^* = \omega e_1 + \frac{1}{2}\omega^2 y_1$, and

$$k(e_2^* + \omega J^{-1}y_2) = (e^{k\omega J} - I)J^{-1}x_2 \quad (6.4)$$

for every k .

Now, $e^{\omega J}$ is an element of the identity component of the pseudorthogonal group of isometries of $\langle \cdot, \cdot \rangle$, and as such can be decomposed into a product of reflections, ordinary rotations, and boosts. With respect to appropriate coordinates, which may be different from our standard choice, a boost will have a matrix of the form

$$\begin{bmatrix} \cosh s & \sinh s \\ \sinh s & \cosh s \end{bmatrix}$$

on some pair of basis vectors, for some $s \in \mathbb{R}$.

If $e^{\omega J}$ is composed only of reflections and ordinary rotations, then the right-hand side of (6.4) is uniformly bounded in k (say, with respect to the positive definite $\langle \cdot, \iota \rangle$) while the left-hand side is unbounded, so $e_2^* + \omega J^{-1}y_2 = 0$. On the other hand, if $e^{\omega J}$ is a pure boost, then the right-hand side grows exponentially in k while the left-hand side grows but linearly, and again we obtain $e_2^* + \omega J^{-1}y_2 = 0$. The Lemma now follows from this and (6.4) for $k = 1$. \square

The proof that item 3 implies item 2 is the same as the relevant part of the proof of (4.3) in [11]. \square

Corollary 6.22 *When in addition $v^* = v_0 = 0$, the following are also equivalent to the three items in Proposition 6.20.*

1. $\dot{\gamma}(0)$ is orthogonal to the orbit $Z_{e^*}n$, where $Z_{e^*} = \exp([e^*, \mathfrak{n}]) \subseteq Z$;
2. $\dot{\gamma}(\omega)$ is orthogonal to the orbit $Z_{e^*}\phi n$.

Proof: Note that under this hypothesis, $x^* = e^*$. Now Lemma 6.21 implies that $J(e^*) = 0$, and this is now equivalent to $z_0 \perp [e^*, \mathfrak{n}]$. Thus the relevant parts of the proof of (4.3) in [11] apply *mutatis mutandis*. \square

We also obtain the following results as in Eberlein [11, (4.4), (4.9)]. Note that we assume that $v^* = v_0 = 0$ in the first, but that this is automatic in the second.

Corollary 6.23 *Let $\phi \in N$ and write $\phi = \exp(a^* + e^*)$ for unique elements $a^* \in \mathfrak{z}$ and $e^* \in \mathfrak{E}$. Let $n \in N$ be given and write $n = \exp(\xi)$ for a unique $\xi \in \mathfrak{n}$. Then the following are equivalent:*

1. *There exists a geodesic γ in N with $\gamma(0) = n$ such that $\phi\gamma(t) = \gamma(t+\omega)$ for all $t \in \mathbb{R}$ and some $\omega > 0$.*
2. *There exists a geodesic γ^* in N with $\gamma^*(0) = 1$, $\dot{\gamma}^*(0)$ is orthogonal to $[e^*, \mathfrak{n}]$, and $\gamma^*(\omega) = \exp([e^*, \xi])\phi$ for some $\omega > 0$.* \square

Corollary 6.24 *Let $1 \neq \phi \in Z$ and γ be any geodesic such that $\gamma(\omega) = \phi\gamma(0)$ for some $\omega > 0$. Then $\phi\gamma(t) = \gamma(t+\omega)$ for all $t \in \mathbb{R}$.* \square

In the flat 2-step nilmanifolds of Theorem 3.15, we can calculate the period spectrum completely.

Theorem 6.25 *If $[\mathfrak{n}, \mathfrak{n}] \subseteq \mathfrak{U}$ and $\mathfrak{E} = \{0\}$, then $\text{spec}_{\wp}(M)$ can be completely calculated from $\log \Gamma$ for any $M = \Gamma \backslash N$.*

Proof: Let ϕ translate a unit-speed geodesic γ by $\omega > 0$. As usual, we may as well assume that $\gamma(0) = 1 \in N$. Write $\log \phi = a^* + v^*$ and $\dot{\gamma}(0) = a_0 + v_0$. From Corollary 4.13 we get $v^* = v(\omega) = \omega v_0$, $z^* = z(\omega) = \omega z_0$, and $u^* = u(\omega) = \omega u_0 + \frac{1}{2}\omega^2 \mathcal{J}v_0$. Note that $\omega^2 \mathcal{J}v_0 = \omega^2 \text{ad}_{v_0}^\dagger(z_0 + v_0) = \omega^2 \text{ad}_{v_0}^\dagger v_0 = \text{ad}_{v^*}^\dagger v^*$. Substituting and rearranging, we obtain

$$\varepsilon\omega^2 = 2\langle u^*, v^* \rangle + \langle z^*, z^* \rangle,$$

where $\pm 1 = \varepsilon = \langle \dot{\gamma}(0), \dot{\gamma}(0) \rangle = 2\langle u_0, v_0 \rangle + \langle z_0, z_0 \rangle$. \square

Thus we see again, as mentioned after Corollary 4.13, just how much these flat, 2-step nilmanifolds are like tori. All periods can be calculated purely from $\log \Gamma \subseteq \mathfrak{n}$, although some will not show up from the tori in the fibration.

Corollary 6.26 $\text{spec}_\varphi(T_B)$ (respectively, T_F) is $\cup_{\mathcal{C}} \wp(\mathcal{C})$ where the union is taken over all those free homotopy classes \mathcal{C} of closed curves in $M = \Gamma \backslash N$ that do not (respectively, do) contain an element in the center of $\Gamma \cong \pi_1(M)$, except for those periods arising only from unit-speed geodesics in M that project to null geodesics in both T_B and T_F . \square

We note that one might consider using this to assign periods to some null geodesics in the tori T_B and T_F .

When the center is nondegenerate, we obtain results similar to Eberlein's [11, (4.5)].

Proposition 6.27 Assume $\mathfrak{U} = \{0\}$. Let $\phi \in N$ and write $\log \phi = z^* + e^*$. Assume ϕ translates the unit-speed geodesic γ by $\omega > 0$. Let z' denote the component of z^* orthogonal to $[e^*, \mathfrak{n}]$. Let $n = \gamma(0)$ and set $\omega^* = |z' + e^*|$. Let $\dot{\gamma}(0) = L_{n^*}(z_0 + e_0)$ and use J , e_1 , and e_2 as in Corollary 4.12 (see also just before Theorem 4.11). Then

1. $|e^*| \leq \omega$. In addition, $\omega < \omega^*$ for timelike (spacelike) geodesics with $\omega z_0 - z'$ timelike (spacelike), and $\omega > \omega^*$ for timelike (spacelike) geodesics with $\omega z_0 - z'$ spacelike (timelike).
2. $\omega = |e^*|$ if and only if $\gamma(t) = \exp(t e^*/|e^*|)$ for all $t \in \mathbb{R}$.
3. $\omega = \omega^*$ if and only if $\omega z_0 - z'$ is null. If moreover $\omega^* z_0 = z'$, then $e_2 = 0$ if and only if
 - (a) $\gamma(t) = n \exp(t(z' + e^*)/\omega^*)$ for all $t \in \mathbb{R}$.
 - (b) $z' = z^* + [e^*, \xi]$ where $\xi = \log n$.

Although ω^* need not be an upper bound for periods as in the Riemannian case, it nonetheless plays a special role among all periods, as seen in item 3 above, and we shall refer to it as the *distinguished* period associated with $\phi \in N$. When the center is definite, for example, we do have $\omega \leq \omega^*$.

Proof: As usual, we may assume that $\gamma(0) = 1 \in N$. Note that this replaces ϕ as given in the statement with $n^{-1}\phi n$ and γ with $n^{-1}\gamma$.

For the first part of item 1, since $|\dot{\gamma}(0)| = 1$ there exists an orthonormal basis of \mathfrak{n} having $\dot{\gamma}(0)$ as a member. (This may well be a different basis from

our usual one.) Fix one such basis, and consider the positive-definite inner product with matrix I on this basis. Let $\|\cdot\|$ denote the norm associated to this positive-definite inner product. By Lemma 6.21, $e^* = \omega e_1$. Then $|e_1| \leq \|e_1\| \leq \|\dot{\gamma}(0)\| = 1$ so $|e^*| = \omega|e_1| \leq \omega$.

For the rest of item 1, we begin with Corollary 4.12 and get

$$\begin{aligned} e(t) &= t e_1 + (e^{tJ} - I) J^{-2} e_2, \\ z(t) &= t (z_0 + \tfrac{1}{2} [e_1, (e^{tJ} + I) J^{-1} e_2]) + z_2(t) + z_3(t). \end{aligned}$$

By Lemma 6.21, $e^* = \omega e_1$ and $e^{\omega J} e_2 = e_2$. Inspecting the formula for $z_2(t)$ in Corollary 4.12, we find $z_2(\omega) = 0$. Thus

$$\begin{aligned} z^* &= z(\omega) = \omega (z_0 + [e_1, J^{-1} e_2]) + z_3(\omega) \\ &= \omega z_0 + [e^*, J^{-1} e_2] + \tfrac{1}{2} \int_0^\omega [e^{sJ} J^{-1} e_2, e^{sJ} e_2] ds. \end{aligned}$$

By item 1 of Corollary 6.22, $z_0 \perp [e^*, \mathfrak{n}]$. Then

$$\langle z', z_0 \rangle = \langle z^*, z_0 \rangle = \omega \langle z_0, z_0 \rangle + \tfrac{1}{2} \int_0^\omega \langle [e^{sJ} J^{-1} e_2, e^{sJ} e_2], z_0 \rangle ds.$$

Recall that J is skewadjoint with respect to $\langle \cdot, \cdot \rangle$ (whence e^{tJ} is an isometry of $\langle \cdot, \cdot \rangle$ for all t), that J commutes with every e^{tJ} (whence so does J^{-1}), and that $Jx = \text{ad}_x^\dagger z_0$. We compute

$$\begin{aligned} \langle [e^{sJ} J^{-1} e_2, e^{sJ} e_2], z_0 \rangle &= -\langle [e^{sJ} e_2, e^{sJ} J^{-1} e_2], z_0 \rangle \\ &= -\langle J^{-1} e^{sJ} e_2, J e^{sJ} e_2 \rangle \\ &= \langle e^{sJ} e_2, e^{sJ} e_2 \rangle \\ &= \langle e_2, e_2 \rangle. \end{aligned}$$

Therefore,

$$\langle z', z_0 \rangle = \omega \langle z_0, z_0 \rangle + \frac{\omega}{2} \langle e_2, e_2 \rangle. \quad (6.5)$$

Now $|\dot{\gamma}(0)| = 1$ so $\varepsilon = \langle z_0, z_0 \rangle + \langle e_1, e_1 \rangle + \langle e_2, e_2 \rangle$, where $\varepsilon = \pm 1$ as usual. Substituting in (6.5) for $\langle e_2, e_2 \rangle$, we obtain

$$\langle z', z_0 \rangle = \frac{\omega}{2} (\varepsilon + \langle z_0, z_0 \rangle) - \frac{\omega}{2} \frac{\langle e^*, e^* \rangle}{\omega^2}$$

so

$$\langle e^*, e^* \rangle - \varepsilon \omega^2 = \omega^2 \langle z_0, z_0 \rangle - 2\omega \langle z', z_0 \rangle.$$

Adding $\langle z', z' \rangle$ to both sides, we get

$$\langle z' + e^*, z' + e^* \rangle - \varepsilon \omega^2 = \langle \omega z_0 - z', \omega z_0 - z' \rangle. \quad (6.6)$$

There are several cases: ε is 1 or -1 and $\omega z_0 - z'$ is timelike, spacelike, or null. If $\omega z_0 - z'$ is null, then $|\langle z' + e^*, z' + e^* \rangle| = \omega^2 > 0$ and $\omega = |z' + e^*|$. If $\varepsilon = 1$ and $\omega z_0 - z'$ is timelike, or if $\varepsilon = -1$ and $\omega z_0 - z'$ is spacelike, then $\varepsilon \langle z' + e^*, z' + e^* \rangle > \omega^2 > 0$ whence $\omega < |z' + e^*|$. If $\varepsilon = 1$ and $\omega z_0 - z'$ is spacelike, or if $\varepsilon = -1$ and $\omega z_0 - z'$ is timelike, then it follows similarly that $\omega > |z' + e^*|$. This completes the proof of item 1.

Now we prove item 2. If γ is as given there, then $\exp(z^* + e^*) = \phi = \gamma(\omega) = \exp(\omega e^*/|e^*|)$ whence $\omega = |e^*|$. Conversely, assume $\omega = |e^*|$ and consider the associated positive-definite inner product $\langle \cdot, \iota \cdot \rangle$. Changing the basis of \mathfrak{E} if necessary, we may assume that \mathfrak{Z} , \mathfrak{E}_1 , and \mathfrak{E}_2 are mutually orthogonal with respect to both $\langle \cdot, \cdot \rangle$ and $\langle \cdot, \iota \cdot \rangle$. Let $\|\cdot\|$ now denote the norm for $\langle \cdot, \iota \cdot \rangle$. Then

$$\|\dot{\gamma}(0)\|^2 = \|z_0\|^2 + \|e_1\|^2 + \|e_2\|^2 \quad (6.7)$$

so $\|\dot{\gamma}(0)\|^2 = \|e_1\|^2$ if and only if $\dot{\gamma}(0) = e_1 = e^*/|e^*|$. But now γ has the same initial data as $\exp(t e^*/|e^*|)$, so by uniqueness they must coincide.

Finally, we prove the last part of item 3; the first part is immediate from the last part of the proof of item 1 above. So assume $\omega^* z_0 - z' = 0$ or $z_0 = z'/\omega^*$. Continue with the immediately previous positive-definite norm $\|\cdot\|$ and basis of \mathfrak{E} . Substituting in (6.7) we get

$$\begin{aligned} \|\dot{\gamma}(0)\|^2 &= \frac{\|z'\|^2}{(\omega^*)^2} + \frac{\|e^*\|^2}{(\omega^*)^2} + \|e_2\|^2 \\ &= \frac{\|z' + e^*\|^2}{(\omega^*)^2} + \|e_2\|^2 \end{aligned}$$

whence

$$\dot{\gamma}(0) = \frac{z' + e^*}{\omega^*}$$

if and only if $e_2 = 0$. □

Corollary 6.28 *Assume the center is nondegenerate. Let $\phi \in N$ with $\phi \notin Z$ and suppose that $z^* \in [e^*, \mathfrak{n}]$. Then*

1. *If ϕ translates a timelike (spacelike) geodesic with z_0 nonspacelike (nontimelike), then ϕ has the unique period $|e^*|$.*

2. Let γ be a unit-speed geodesic in N with $\gamma(0) = n = \exp(\xi)$ for a unique $\xi \in \mathfrak{n}$. Then ϕ translates γ by the unique period $|e^*| > 0$ if and only if $[\xi, e^*] = z^*$ and $\gamma(t) = n \exp(t e^*/|e^*|)$ for all $t \in \mathbb{R}$.

In particular, this applies to all noncentral $\phi \in N$ if \mathfrak{n} is nonsingular. \square

The proof follows that of [11, (4.6)] *mutatis mutandis* and we omit the details. We do note that from item 2 of Proposition 6.27, using Lemma 1.1, we obtain $\exp(e^* + \frac{1}{2}[\xi, e^*]) = n \exp(e^*) = \gamma(|e^*|) = \phi n = \exp(z^* + e^*) \exp(\xi) = \exp(z^* + e^* + \frac{1}{2}[e^*, \xi])$, thus avoiding the use of item 3 here. Anent the last comment, note that if \mathfrak{n} is nonsingular then in fact $z_0 = 0$ in item 1, because $z_0 \perp [e^*, \mathfrak{n}] = \mathfrak{z}$.

In view of the comment following Proposition 6.27 and Corollary 6.28, the following definitions make sense at least for N with a nondegenerate center.

Definition 6.29 Let \mathcal{C} denote either a nontrivial, free homotopy class of closed curves in $\Gamma \backslash N$ or the corresponding conjugacy class in Γ . We define $\wp^*(\mathcal{C})$ to be the distinguished periods of periodic unit-speed geodesics that belong to \mathcal{C} .

Definition 6.30 The distinguished period spectrum of $\Gamma \backslash N$ is the set

$$\mathcal{D}_{\text{spec}}(\Gamma \backslash N) = \bigcup_{\mathcal{C}} \wp^*(\mathcal{C}),$$

where the union is taken over all nontrivial, free homotopy classes of closed curves in $\Gamma \backslash N$.

Then as an immediate consequence of the preceding corollary, we get this result.

Corollary 6.31 Assume the center is nondegenerate. If \mathfrak{n} is nonsingular, then $\text{spec}_{\wp}(T_B)$ (respectively, T_F) is precisely the period spectrum (respectively, the distinguished period spectrum) of those free homotopy classes \mathcal{C} of closed curves in $M = \Gamma \backslash N$ that do not (respectively, do) contain an element in the center of $\Gamma \cong \pi_1(M)$, except for those periods arising only from unit-speed geodesics in M that project to null geodesics in both T_B and T_F . \square

7 Constructing Lie algebras of pH -type

Let us suppose that $\mathfrak{n} = \mathfrak{z} \oplus \mathfrak{v} = \mathfrak{U} \oplus \mathfrak{Z} \oplus \mathfrak{V} \oplus \mathfrak{E}$ is a vector space (not a Lie algebra), for which we have a basis consisting of

$$\begin{aligned} \{u_i\} &\text{ a basis for } \mathfrak{U}, & \{z_\alpha\} &\text{ a basis for } \mathfrak{Z}, \\ \{v_i\} &\text{ a basis for } \mathfrak{V}, & \{e_a\} &\text{ a basis for } \mathfrak{E}. \end{aligned}$$

Assume that there exists a nontrivial inner product $\langle \cdot, \cdot \rangle$ on \mathfrak{n} which is given with respect this basis by

$$\langle u_i, v_i \rangle = 1, \quad \langle z_\alpha, z_\alpha \rangle = \varepsilon_\alpha, \quad \langle e_a, e_a \rangle = \bar{\varepsilon}_a.$$

Here, each ε -symbol is ± 1 independently, making the bases for \mathfrak{Z} and \mathfrak{E} orthonormal.

Then we define an involution $\iota: \mathfrak{n} \rightarrow \mathfrak{n}$ with respect to this basis by setting:

$$\iota(u_i) = v_i, \quad \iota(v_i) = u_i, \quad \iota(z_\alpha) = \varepsilon_\alpha z_\alpha, \quad \iota(e_a) = \bar{\varepsilon}_a e_a.$$

Thus $\iota(\mathfrak{U}) = \mathfrak{V}$, $\iota(\mathfrak{V}) = \mathfrak{U}$, $\iota(\mathfrak{Z}) = \mathfrak{Z}$, $\iota(\mathfrak{E}) = \mathfrak{E}$, and $\iota^2 = I$. Moreover, ι is self-adjoint with respect to the inner product: $\langle \iota x, y \rangle = \langle x, \iota y \rangle$ for every $x, y \in \mathfrak{n}$. Therefore ι is an isometry of \mathfrak{n} . Note that $\langle x, \iota x \rangle = 0$ if and only if $x = 0$.

Let us now also assume that there is given a linear mapping

$$j: \mathfrak{U} \oplus \mathfrak{Z} \rightarrow \text{End}(\mathfrak{V} \oplus \mathfrak{E})$$

satisfying the conditions

$$\langle j(a)x, \iota j(a)x \rangle = \langle a, \iota a \rangle \langle x, \iota x \rangle \tag{7.1}$$

$$j(a)^2 = -\langle a, \iota a \rangle I \tag{7.2}$$

for every $a \in \mathfrak{U} \oplus \mathfrak{Z}$ and $x \in \mathfrak{V} \oplus \mathfrak{E}$.

Then there exists on \mathfrak{n} a canonical Lie algebra structure of pH -type, obtained as follows.

First, we note that polarizing (7.1) and (7.2) yields the relations

$$\langle j(a)x, \iota j(b)x \rangle = \langle a, \iota b \rangle \langle x, \iota x \rangle \tag{7.3}$$

$$\langle j(a)x, \iota j(a)y \rangle = \langle a, \iota a \rangle \langle x, \iota y \rangle \tag{7.4}$$

for every $a, b \in \mathfrak{U} \oplus \mathfrak{Z}$ and $x, y \in \mathfrak{V} \oplus \mathfrak{E}$. Then it is easy to prove that the map j is ι -skewsymmetric:

$$\langle j(a)x, \iota y \rangle + \langle \iota x, j(a)y \rangle = 0$$

for every $a \in \mathfrak{U} \oplus \mathfrak{Z}$ and $x, y \in \mathfrak{V} \oplus \mathfrak{E}$. Indeed, if $a = 0$ this is trivial, and if $a \neq 0$ then from (7.1), (7.2), and (7.4), we get

$$\begin{aligned} \langle j(a)x, \iota y \rangle &= \langle j(a)x, \iota j(a)^2 (-\langle a, \iota a \rangle^{-1} y) \rangle \\ &= -\langle a, \iota a \rangle^{-1} \langle j(a)x, \iota j(a) (j(a)y) \rangle \\ &= -\langle a, \iota a \rangle^{-1} \langle a, \iota a \rangle \langle x, \iota j(a)y \rangle \\ &= -\langle \iota x, j(a)y \rangle. \end{aligned}$$

Now we define a bilinear map

$$[\cdot, \cdot] : (\mathfrak{V} \oplus \mathfrak{E}) \oplus (\mathfrak{V} \oplus \mathfrak{E}) \longrightarrow \mathfrak{U} \oplus \mathfrak{Z}$$

as follows:

$$\langle [x, y], \iota a \rangle = \langle j(a)x, \iota y \rangle$$

for every $x, y \in \mathfrak{V} \oplus \mathfrak{E}$ and $a \in \mathfrak{U} \oplus \mathfrak{Z}$. We remark that $[x, y] \in \mathfrak{U} \oplus \mathfrak{Z}$ is well defined.

Proposition 7.1 *This map $[\cdot, \cdot]$ is skewsymmetric: $[x, y] = -[y, x]$ for every $x, y \in \mathfrak{V} \oplus \mathfrak{E}$.*

Proof: We compute:

$$\begin{aligned} \langle [x, y], \iota a \rangle &= \langle j(a)x, \iota y \rangle \\ &= -\langle \iota x, j(a)y \rangle \\ &= -\langle [y, x], \iota a \rangle. \end{aligned}$$

□

We can now define a Lie algebra structure on \mathfrak{n} by extending this map to all of \mathfrak{n} via

$$[a + x, b + y] = [x, y] \in \mathfrak{U} \oplus \mathfrak{Z}$$

for every $a, b \in \mathfrak{U} \oplus \mathfrak{Z}$, and $x, y \in \mathfrak{V} \oplus \mathfrak{E}$. Clearly, we obtain

Theorem 7.2 *Endowed with this structure, \mathfrak{n} is a 2-step nilpotent Lie algebra of pH-type with center $\mathfrak{U} \oplus \mathfrak{Z}$.*

□

We illustrate this construction with two simple examples.

Example 7.3 Assume $\mathfrak{U} = \mathfrak{V} = \{0\}$, $\dim \mathfrak{Z} = 1$, and $\dim \mathfrak{E} = 2$, so that $\dim \mathfrak{n} = 3$. Choose bases $\{z\}$ of \mathfrak{Z} and $\{e_1, e_2\}$ of \mathfrak{E} , and nontrivial inner products

$$\varepsilon = \langle z, z \rangle, \quad \bar{\varepsilon}_i = \langle e_i, e_i \rangle.$$

Suppose that the mapping

$$j : \mathfrak{Z} \rightarrow \text{End}(\mathfrak{E})$$

is given with respect to the basis $\{e_1, e_2\}$ as

$$j(z) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

Then

$$\langle [e_1, e_2], \iota z \rangle = \varepsilon \langle [e_1, e_2], z \rangle,$$

and, on the other hand,

$$\langle j(z)e_1, \iota e_2 \rangle = \bar{\varepsilon}_2 \langle j(z)e_1, e_2 \rangle = \bar{\varepsilon}_2 \langle e_2, e_2 \rangle = 1.$$

Hence, $\varepsilon \langle [e_1, e_2], z \rangle = 1$ so $[e_1, e_2] = z$.

Therefore, \mathfrak{n} is the 3-dimensional Lie algebra with structure equation $[e_1, e_2] = z$; that is, the 3-dimensional Heisenberg algebra with non-null center.

Example 7.4 Assume $\dim \mathfrak{U} = \dim \mathfrak{V} = 1$, $\mathfrak{Z} = \{0\}$ and $\dim \mathfrak{E} = 1$, so that $\dim \mathfrak{n} = 3$ again. Choose bases $\{u\}$ for \mathfrak{U} , $\{v\}$ for \mathfrak{V} , and $\{e\}$ for \mathfrak{E} , and nontrivial inner products

$$\langle u, v \rangle = 1, \quad \langle e, e \rangle = \bar{\varepsilon}.$$

Suppose that the mapping

$$j : \mathfrak{U} \rightarrow \text{End}(\mathfrak{V} \oplus \mathfrak{E})$$

is given with respect to the basis $\{v, e\}$ as

$$j(u) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

Then

$$\langle [v, e], \iota u \rangle = \langle [v, e], v \rangle,$$

and, on the other hand,

$$\langle j(u)e, \nu \rangle = \langle j(u)e, u \rangle = \langle v, u \rangle = 1.$$

Hence, $\langle [v, e], v \rangle = 1$ so $[v, e] = u$.

Therefore, \mathfrak{n} is the 3-dimensional Lie algebra with structure equation $[v, e] = u$; that is, the 3-dimensional Heisenberg algebra with null center.

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